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THE MOTION OF A SATELLITE OF THE MOON

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THE MOTION OF A SATELLITE OF THE MOON

by

Giorgio E. O. Giacaglia, James P. Murphy,
and Theodore L. Felsentreger

in collaboration with

E. Myles Standish, Jr., and Carmelo E. Velez

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Goddard Space Flight Center
Greenbelt, Maryland

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SUMMARY

An analytical solution to the problem of the motion of a satellite of the moon is presented. Perturbations of short period and of intermediate period are derived through the application of the von Zeipel method. Long period perturbations are obtained through the use of elliptic integrals.

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1. Introduction

The motion of a satellite of the moon, or lunar orbiter, is analyzed. The solution is developed in powers of 10^{-2}. Thus, a first order quantity is of order 10^{-2}, a second order quantity is of the order 10^{-4}, and so on. The Hamiltonian for the "main problem" consists of zero, first, and second order quantities. The higher order Hamiltonian is of third order and consists of two parts: the first part contains terms generated by coupling of lower order terms and the second part consists of terms added by considering further perturbing forces, such as solar radiation pressure, physical libration, non-sphericity of the Earth's potential field, the attraction of the sun, etc. Additional terms are also produced by considering the eccentricity and inclination of the moon's orbit.

In order to retain the relative orders of the disturbing forces, it is necessary to restrict the semi-major axis of the orbiter to about four moon radii or less. Further, certain restrictions must be made on the eccentricity and inclination in order not to invalidate the solution. Therefore, the following assumptions on the semi-major axis, eccentricity, and inclination are made:

$$a \leq 4 \text{ moon radii}$$

$$.01 < e < .75$$

$$|\sin I| > .01 .$$

The small parameter of first order is $n_{\text{☾}}^*$, which is the mean motion of the moon's mean longitude. The small parameters of second order are J_2, J_{22}, and $(n_{\text{☾}}/n)^2$ where J_2 and J_{22} define the principal part of the oblateness of the moon and $(n_{\text{☾}}/n)^2$ is the square of the ratio of the mean motion of the moon to that of the orbiter. The small parameters of third order are J_3, J_4, J_5, $j_2\ (n_{\text{☾}}/n)^2$, $(n_{\oplus}/n)^2$, $(n_{\text{☾}}/n)^2 \sin(i_{\text{☾}}/2)$, $(n_{\text{☾}}/n)^2\, e_{\text{☾}}$, σ, $(n_{\text{☾}}/n)^3$, and $\alpha n_{\odot}/n$. The quantities J_3, J_4, and J_5 are higher oblateness

parameters of the moon, and j_2 is the principal term in the oblateness of the earth. The quantity $(n_\oplus/n)^2$ is the square of the ratio of the mean motion of the earth to that of the orbiter. The fact that the moon's orbital plane is inclined to its equator and the fact that the moon's orbit about the earth is elliptical give rise to the two small parameters of third order $(n_☾/n)^2 \sin(i_☾/2)$ and $(n_☾/n)^2 e_☾$, respectively. The radiation pressure gives rise to the third order parameter σ. $(n_☾/n)^3$ is the cube of the ratio of the mean motion of the moon to that of the orbiter. Finally, $\alpha n_\odot/n$ is the correction due to physical libration.

For the moon the values of J_2 and J_{22} published by Jeffreys (Reference 5) are adopted, i.e.

$$J_2 = 2.41 \times 10^{-4}$$

$$J_{22} = 0.21 \times 10^{-4}$$

The longest meridian of the moon contains the line joining the centers of mass of the earth and the moon. The right-handed, rotating, selenocentric coordinate systems adopted for this problem will then be as follows: The z-axis is the rotational axis of the moon, and the xy-plane is the moon's equatorial plane. The x-axis passes through the moon's longest meridian, and is assumed to rotate with the motion $n_☾^*$.

For a semimajor axis of 4 moon radii,

$$\left(\frac{n_☾}{n}\right)^2 \simeq 4.8 \times 10^{-4}$$

$$\left(\frac{n_\oplus}{n}\right)^2 \simeq 2.7 \times 10^{-4}$$

so that the oblateness terms in the lunar potential and the earth perturbations are both about of the same order. The largest oblateness term of the earth is then of the order of 10^{-6}. The perturbation of the sun is of third order.

Another perturbation to be considered is the effective radiation pressure of the sun, whose strength is about 1×10^{-4} dyne/cm^2. If the area-mass ratio of the orbiter is 1.5×10^{-1} cm^2/g, then the disturbing acceleration due to radiation pressure is also of third order.

It is plain to see that a theory which includes the moon's oblateness and the earth perturbations should also include the solar perturbations in the "main problem" if the semi-major axis is above four moon radii.

2. Equations of Motion

The first step is to determine the equations of motion for the gravitational fields of the moon as a primary and the earth and sun as perturbations.

The following notation will be used:

subscript	0	:	moon
"	1	:	orbiter
"	2	:	earth
"	3	:	sun.

In an inertial system the equations of motion are

$$m_j \ddot{\boldsymbol{\rho}}_j = \mathrm{grad}_{\boldsymbol{\rho}_j} U \quad (j = 0, 1, 2, 3)$$

where

$$U = k^2 \left(\frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_0 m_3}{r_{03}} + \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right)$$

and $\boldsymbol{\rho}_j$ is the radius vector of any one of the 4 bodies. If ξ_j, η_j, ζ_j are the rectangular inertial coordinates of one of the bodies, then the equations of motion can also be written

$$m_j \ddot{\xi}_j = \frac{\partial U}{\partial \xi_j} \quad (j = 0, 1, 2, 3).$$

Similar expressions hold for η_j and ζ_j.

It is now convenient to refer the orbiter to a moon centered system, the moon to an earth centered system and the sun to a system whose origin is at the center of mass of the earth-moon system. Therefore

$$x_1 = \xi_1 - \xi_0$$

$$x_0 = \xi_0 - \xi_2$$

$$x_3 = \xi_3 - \xi_G = \xi_3 - \frac{m_0 \xi_0 + m_2 \xi_2}{m_0 + m_2} .$$

Similar expressions hold for y_j and z_j.

The equations of motion must be transformed accordingly. The partials in Eqs. (2) are then computed with respect to the new variables by making use of

$$\frac{\partial U}{\partial \xi_k} = \sum_{j=0}^{3} \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial \xi_k} \quad (k = 0, 1, 2, 3)$$

It follows that

$$\frac{\partial U}{\partial \xi_0} = \frac{\partial U}{\partial x_0} - \frac{\partial U}{\partial x_1} - \frac{m_0}{m_0 + m_2} \frac{\partial U}{\partial x_3}$$

$$\frac{\partial U}{\partial \xi_1} = \frac{\partial U}{\partial x_1}$$

$$\frac{\partial U}{\partial \xi_2} = - \frac{\partial U}{\partial x_0} - \frac{m_2}{m_0 + m_2} \frac{\partial U}{\partial x_3}$$

$$\frac{\partial U}{\partial \xi_3} = \frac{\partial U}{\partial x_3} .$$

Since

$$\ddot{x}_1 = \ddot{\xi}_1 - \ddot{\xi}_0 ,$$

it follows that

$$\ddot{x}_1 = \left(\frac{1}{m_1} + \frac{1}{m_0}\right) \frac{\partial U}{\partial x_1} - \frac{1}{m_0} \frac{\partial U}{\partial x_0} + \frac{1}{m_0 + m_2} \frac{\partial U}{\partial x_3} .$$

The force function U must now be expressed in terms of the new coordinates. We have

$$r_{01}^2 = (\xi_1 - \xi_0)^2 + (\eta_1 - \eta_0)^2 + (\zeta_1 - \zeta_0)^2 = x_1^2 + y_1^2 + z_1^2 = r_1^2$$

$$r_{02}^2 = (\xi_0 - \xi_2)^2 + \ldots = x_0^2 + \ldots = r_0^2$$

$$r_{03}^2 = (\xi_3 - \xi_0)^2 + \ldots = \left(x_3 - \frac{m_2}{m_0 + m_2} x_0\right)^2 + \ldots =$$

$$= x_3^2 + y_3^2 + z_3^2 + \left(\frac{m_2}{m_0 + m_2}\right)^2 (x_0^2 + y_0^2 + z_0^2) - \frac{2m_2}{m_0 + m_2} (x_0 x_3 + y_0 y_3 + z_0 z_3)$$

$$= r_3^2 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_0^2 - \frac{2m_2}{m_0 + m_2} \mathbf{r}_0 \cdot \mathbf{r}_3$$

$$r_{12}^2 = (\xi_1 - \xi_2)^2 + \ldots = (x_1 + x_0)^2 + \ldots = r_1^2 + r_0^2 + 2\mathbf{r}_1 \cdot \mathbf{r}_0$$

$$r_{13}^2 = (\xi_1 - \xi_3)^2 + \ldots = \left[(x_1 - x_3) + \frac{m_2}{m_0 + m_2} x_0\right]^2 + \ldots =$$

$$= r_1^2 + r_3^2 - 2\mathbf{r}_1 \cdot \mathbf{r}_3 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_0^2 + \frac{2m_2}{m_0 + m_2} (\mathbf{r}_0 \cdot \mathbf{r}_1 - \mathbf{r}_0 \cdot \mathbf{r}_3)$$

$$r_{23}^2 = (\xi_2 - \xi_3)^2 + \ldots = \left(-x_3 - \frac{m_0}{m_0 + m_2} x_0\right)^2 + \ldots =$$

$$= r_3^2 + \left(\frac{m_0}{m_0 + m_2}\right)^2 r_0^2 + \frac{2m_0}{m_0 + m_2} \mathbf{r}_0 \cdot \mathbf{r}_3 \; .$$

Since U contains the inverses of these radii, note that

$$\frac{1}{r_{01}} = \frac{1}{r_1}$$

$$\frac{1}{r_{02}} = \frac{1}{r_0}$$

$$\frac{1}{r_{03}} = \frac{1}{r_3}\left\{1 + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 - \frac{2m_2}{m_0 + m_2}\ \frac{r_0}{r_3}\cos S_{03}\right\}^{-1/2}$$

$$\frac{1}{r_{12}} = \frac{1}{r_0}\left\{1 + \left(\frac{r_1}{r_0}\right)^2 + 2\,\frac{r_1}{r_0}\cos S_{10}\right\}^{-1/2}$$

$$\frac{1}{r_{13}} = \frac{1}{r_3}\left\{1 + \left(\frac{r_1}{r_3}\right)^2 - 2\,\frac{r_1}{r_3}\cos S_{13} + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 + \right.$$

$$\left. + \frac{2m_2}{m_0 + m_2}\left(\frac{r_0\ r_1}{r_3^2}\cos S_{10} - \frac{r_0}{r_3}\cos S_{03}\right)\right\}^{-1/2}$$

$$\frac{1}{r_{23}} = \frac{1}{r_3}\left\{1 + \left(\frac{m_0}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 + \frac{2m_0}{m_0 + m_2}\ \frac{r_0}{r_3}\cos S_{03}\right\}^{-1/2} .$$

The angles used are shown in Figure 1. For obvious reasons the angle S_{03} is expressed in terms of S'_{03}, as follows:

Since $$\cos S'_{03} = \frac{\mathbf{r}_0 \cdot \mathbf{r}'_3}{r_0\ r'_3}\ ,$$

and $$\frac{m_0}{m_0 + m_2}\,\mathbf{r}_0 + \mathbf{r}_3 = \mathbf{r}'_3\ ,$$

then $$\cos S'_{03} = \frac{\frac{m_0}{m_0 + m_2}\, r_0^2 + r_0\ r_3 \cos S_{03}}{r_0\ r'_3} =$$

$$= \frac{r_0}{r'_3}\cdot\frac{m_0}{m_0 + m_2} + \frac{r_3}{r'_3}\cos S_{03}$$

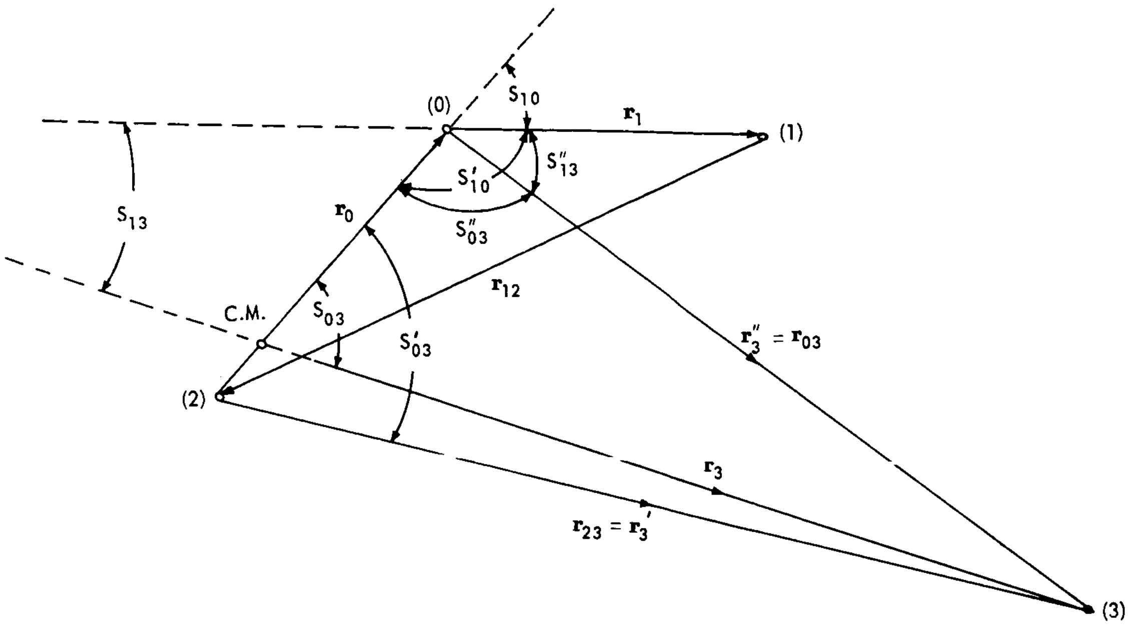

Figure 1

or, finally

$$\cos S_{03} = \frac{r'_3}{r_3}\left(\cos S'_{03} - \frac{r_0}{r'_3}\,\frac{m_0}{m_0 + m_2}\right) .$$

Certainly, to third order, $r'_3 \simeq r_3$, so that

$$\cos S_{03} = \cos S'_{03} - \frac{r_0}{r_3}\,\frac{m_0}{m_0 + m_2} .$$

On the other hand, S_{10} may be replaced by $180^\circ - S'_{10}$ where S'_{10} is the selenocentric elongation between the earth and the orbiter.

3. Oblateness Terms

If the plane of reference is the lunar equatorial plane, then the disturbing force per unit mass may be written as

$$U_{OBL.} = \frac{k^2 m_0}{r_1} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_☾}{r_1}\right)^n J_{nm} P_{nm} (\sin \beta) \cos m (\lambda - \lambda_{nm}) .$$

The only terms to be included are J_{22}, $J_{20} = -J_2$, $J_{30} = -J_3$, $J_{40} = -J_4$, and $J_{5,0} = -J_5$. Therefore,

$$U_{OBL.} = -\frac{\mu_0}{r_1} \left\{ \sum_{n=2}^{5} \left(\frac{R_☾}{r_1}\right)^n J_n P_n (\sin \beta) \right.$$

$$\left. - \left(\frac{R_☾}{r_1}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2 (\lambda' - \lambda_{22}) \right\} .$$

The angle β is the latitude of the orbiter with respect to the equator of the moon, λ' its longitude reckoned from any fixed direction and λ_{22} the longitude of the moon's longest meridian from the same fixed direction. Note that λ' and β can be expressed in terms of the coordinates x_1, y_1, z_1 of the orbiter. However, λ_{22} will contain the time explicitly, since

$$\lambda_{22} = \lambda_{22}(0) + \gamma_☾ t$$

where $\gamma_☾$ is the frequency of rotation of the moon around its axis. Further, if we neglect the physical libration of the moon, the longest meridian is always pointing toward the earth and $\gamma_☾ \simeq n_☾^*$.

4. Canonical Equations and Gravitational Terms

If we choose as canonical variables the Delaunay set

$L = \sqrt{\mu_0 a}$, ℓ = mean anomaly,

$G = L\sqrt{1 - e^2}$, ω = argument of pericenter,

$H = G \cos I$, Ω = longitude of ascending node,

where a, e, and I are the semi-major axis, eccentricity, and inclination respectively, and where $\mu_0 = k^2 m_0$ where k is the Gaussian constant and m_0 is the mass of the moon, then the equations of motion become

$$\dot{L} = \frac{\partial \tilde{F}}{\partial l}, \quad \dot{G} = \frac{\partial \tilde{F}}{\partial \omega}, \quad \dot{H} = \frac{\partial \tilde{F}}{\partial \Omega},$$

$$\dot{l} = -\frac{\partial \tilde{F}}{\partial L}, \quad \dot{\omega} = -\frac{\partial \tilde{F}}{\partial G}, \quad \dot{\Omega} = -\frac{\partial \tilde{F}}{\partial H},$$

where

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + U_{GRAV.} + U_{OBL.}$$

and where $U_{GRAV.}$ is a function which has to satisfy the equation

$$\ddot{x}_1 = \frac{\partial}{\partial x_1}\left(U_{GRAV.} + \frac{\mu_0}{r_1}\right)$$

The force function U depends on the new variables x through the relations

$$r_{01}^2 = x_1^2 + y_1^2 + z_1^2$$

$$r_{02}^2 = x_0^2 + y_0^2 + z_0^2$$

$$r_{03}^2 = \left(x_3 - \frac{m_2}{m_0 + m_2} x_0\right)^2 + \ldots$$

$$r_{12}^2 = (x_1 + x_0)^2 + \ldots$$

$$r_{13}^2 = \left[(x_1 - x_3) + \frac{m_2}{m_0 + m_2} x_0\right]^2 + \ldots$$

$$r_{23}^2 = \left(- x_3 - \frac{m_0}{m_0 + m_2} x_0\right)^2 + \ldots\ldots$$

Then,

$$\frac{\partial U}{\partial x_0} = k^2 \left\{ - m_0 m_2 \frac{x_0}{r_{02}^3} - m_0 m_3 \frac{x_3 - \frac{m_2}{m_0 + m_2} x_0}{r_{03}^3} \left(- \frac{m_2}{m_0 + m_2}\right) \right.$$

$$- m_1 m_2 \frac{x_1 + x_0}{r_{12}^3} - m_1 m_3 \frac{x_1 - x_3 + \frac{m_2}{m_0 + m_2} x_0}{r_{13}^3} \left(\frac{m_2}{m_0 + m_2}\right)$$

$$- m_2\, m_3\, \frac{- x_3 - \dfrac{m_0}{m_0 + m_2}\, x_0}{r_{23}^3} \left(- \frac{m_0}{m_0 + m_2}\right)\Bigg\} =$$

$$= \frac{\partial}{\partial x_1} \Bigg\{- k^2\, m_0\, m_2\, \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{02}^3} + k^2\, \frac{m_0\, m_2\, m_3}{m_0 + m_2}\, \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{03}^3}$$

$$- k^2 \frac{m_0\, m_2^2\, m_3}{(m_0 + m_2)^2}\, \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{03}^3} + k^2\, m_1\, m_2\, \frac{1}{r_{12}}$$

$$+ k^2\, \frac{m_1\, m_2\, m_3}{m_0 + m_2}\, \frac{1}{r_{13}} - k^2\, \frac{m_0\, m_2\, m_3}{m_0 + m_2}\, \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{23}^3}$$

$$- k^2\, \frac{m_0^2\, m_2\, m_3}{(m_0 + m_2)^2}\, \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{23}^3}\Bigg\} ,$$

and

$$\frac{\partial U}{\partial x_3} = k^2 \Bigg\{- m_0\, m_3\, \frac{x_3 - \dfrac{m_2}{m_0 + m_2}\, x_0}{r_{03}^3} + m_1\, m_3\, \frac{x_1 - x_3 + \dfrac{m_2}{m_0 + m_2}\, x_0}{r_{13}^3}$$

$$+ m_2\, m_3\, \frac{- x_3 - \dfrac{m_0}{m_0 + m_2}\, x_0}{r_{23}^3}\Bigg\} =$$

$$= \frac{\partial}{\partial x_1} \left\{ - k^2 m_0 m_3 \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{03}^3} + k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{03}^3} \right.$$

$$\left. - k^2 m_1 m_3 \frac{1}{r_{13}} - k^2 m_2 m_3 \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{23}^3} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{23}^3} \right\} .$$

Then,

$$\ddot{x}_1 = \frac{m_1 + m_0}{m_1 m_0} \frac{\partial}{\partial x_1} \left\{ k^2 \frac{m_0 m_1}{r_{01}} + k^2 m_1 m_2 \frac{1}{r_{12}} + k^2 m_1 m_3 \frac{1}{r_{13}} \right\}$$

$$- \frac{1}{m_0} \frac{\partial}{\partial x_1} \left\{ - k^2 m_0 m_2 \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{02}^3} + k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{03}^3} \right.$$

$$- k^2 \frac{m_0 m_2^2 m_3}{(m_0 + m_2)^2} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{03}^3} + k^2 m_1 m_2 \frac{1}{r_{12}}$$

$$+ k^2 \frac{m_1 m_2 m_3}{m_0 + m_2} \frac{1}{r_{13}} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{23}^3}$$

$$\left. - k^2 \frac{m_0^2 m_2 m_3}{(m_0 + m_2)^2} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{23}^3} \right\} +$$

$$+ \frac{1}{m_0 + m_2} \frac{\partial}{\partial x_1} \left\{ - k^2 m_0 m_3 \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{03}^3} + k^2 \frac{m_0 m_2 m_3}{(m_0 + m_2)} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{03}^3} \right.$$

$$- k^2 m_1 m_3 \frac{1}{r_{13}} - k^2 m_2 m_3 \frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{r_{23}^3}$$

$$\left. - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{23}^3} \right\} =$$

$$= \frac{\partial}{\partial x_1} \left\{ k^2 \frac{m_0 + m_1}{r_{01}} + k^2 \frac{m_2}{r_{12}} + k^2 \frac{m_3}{r_{13}} + k^2 m_2 \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{02}^3} \right.$$

$$\left. - k^2 m_3 \mathbf{r}_1 \cdot \mathbf{r}_3'' \frac{1}{r_{03}^3} \right\} .$$

Therefore,

$$U_{\text{GRAV.}} = \frac{k^2 m_1}{r_{01}} + \frac{k^2 m_2}{r_{12}} + \frac{k^2 m_3}{r_{13}} + k^2 m_2 \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_{02}^3}$$

$$- k^2 m_3 \frac{\mathbf{r}_1 \cdot \mathbf{r}_3''}{r_{03}^3} ,$$

where $$\mathbf{r}_3'' = \mathbf{r}_3 - \frac{m_2}{m_0 + m_2} \mathbf{r}_0$$

and where

$r_{01} = r_1$

$r_{02} = r_0$

r_{13} = series in Legendre Polynomials

r_{03} = series in Legendre Polynomials.

The term $k^2 m_1/r_{01}$ can, of course, be neglected. Finally,

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_{12}} + \frac{\mu_3}{r_{13}} + \mu_2 \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_0^3} - \frac{\mu_3 \mathbf{r}_1 \cdot \mathbf{r}_3''}{r_{03}^3}$$

$$+ \frac{\mu_0}{r_1} \left\{ - \sum_{n=2}^{5} \left(\frac{R_{\mathbb{C}}}{r_1}\right)^n J_n P_n (\sin\beta) + \left(\frac{R_{\mathbb{C}}}{r_1}\right)^2 J_{22} P_{22} (\sin\beta) \cos 2 (\lambda' - \lambda_{22}) \right\}$$

Now

$$\frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_0^3} = \frac{-r_0 r_1 \cos S'_{10}}{r_0^3} = -\frac{r_1}{r_0^2} \cos S'_{10}$$

and

$$\frac{1}{r_{12}} = \frac{1}{r_0} + \frac{r_1}{r_0^2} \cos S'_{10} + \frac{1}{r_0} \sum_{p=2}^{\infty} P_p (\cos S'_{10}) \left(\frac{r_1}{r_0}\right)^p .$$

Then

$$\frac{\mu_2}{r_{12}} + \mu_2 \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_0^3} = \frac{\mu_2}{r_0} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p (\cos S'_{10})$$

and since μ_2/r_0 is independent of the position of the orbiter,

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p (\cos S'_{10}) + \frac{\mu_3}{r_{13}} - \mu_3 \frac{\mathbf{r}_1 \cdot \mathbf{r}_3''}{r_{03}^3}$$

$$+ \frac{\mu_0}{r_1} \left\{ - \sum_{n=2}^{5} \left(\frac{R_{\mathbb{C}}}{r_1}\right)^n J_n P_n (\sin\beta) + \left(\frac{R_{\mathbb{C}}}{r_1}\right)^2 J_{22} P_{22} (\sin\beta) \cos 2 (\lambda' - \lambda_{22}) \right\}.$$

A further simplification can be made if one considers the following expansion:

$$r_{13}^2 = \left[x_1 - \left(x_3 - \frac{m_2}{m_0 + m_2} x_0\right)\right]^2 + \ldots =$$

$$= r_1^2 + r_{03}^2 - 2 r_1 r_{03} \cos S_{13}'' = r_{03}^2 \left[1 + \left(\frac{r_1}{r_{03}}\right)^2 - 2 \frac{r_1}{r_{03}} \cos S_{13}''\right]$$

$$\frac{1}{r_{13}} = \frac{1}{r_{03}} \sum_{p=0}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p (\cos S_{13}'') =$$

$$= \frac{1}{r_{03}} + \frac{r_1 \cos S_{13}''}{r_{03}^2} + \frac{1}{r_{03}} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p (\cos S_{13}'').$$

Also,

$$\frac{\mathbf{r}_1 \cdot \mathbf{r}_3''}{r_{03}^3} = \frac{r_1 r_{03} \cos S_{13}''}{r_{03}^3} = \frac{r_1 \cos S_{13}''}{r_{03}^2}.$$

Since r_{03} is independent of x_1, y_1, z_1,

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p (\cos S_{10}')$$

$$+ \frac{\mu_3}{r_{03}} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p (\cos S_{13}'')$$

$$+ \frac{\mu_0}{r_1} \left\{- \sum_{n=2}^{5} \left(\frac{R_c}{r_1}\right)^n J_n P_n (\sin \beta) + \left(\frac{R_c}{r_1}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2 (\lambda' - \lambda_{22})\right\}$$

From now on the subscript 1 is omitted.

5. Analysis of Terms

Next the order of magnitude of the gravitational terms and the oblateness terms is computed.

Consider the factor

$$\frac{\mu_0}{r^3} = \frac{\mu_0}{a^3 (1 - e^2)^3} + \text{periodic terms in f}$$

where

$$n^2 a^3 = \mu_0 ,$$

so that

$$\frac{\mu_0}{a^3} = n^2 .$$

Then

$$\frac{\mu_0}{a^3 (1 - e^2)^3} R_{\mathfrak{C}}^2 = n^2 \frac{R_{\mathfrak{C}}^2}{(1 - e^2)^3} = \frac{\mu_0^2}{L^2 (1 - e^2)^3} \left(\frac{R_{\mathfrak{C}}}{a}\right)^2$$

$$= \frac{\mu_0^2}{2L^2 (1 - e^2)^3} \left\{2 \left(\frac{R_{\mathfrak{C}}}{a}\right)^2\right\} .$$

The relative magnitude of the greatest oblateness terms are then given by

$$2\left(\frac{R_{\mathfrak{C}}}{a}\right)^2 J_2 \text{ and } 2\left(\frac{R_{\mathfrak{C}}}{a}\right)^2 J_{22} .$$

Consider a very low orbiter for which $a = 2R_{\mathfrak{C}}$. Then

$$2\left(\frac{R_{☾}}{a}\right)^2 J_2 \simeq 2 \times \frac{1}{4} \times 2 \times 10^{-4} = 10^{-4}$$

$$2\left(\frac{R_{☾}}{a}\right)^2 J_{22} \simeq 2 \times \frac{1}{4} \times 0.2 \times 10^{-4} = 10^{-5}.$$

Consider the earth's perturbation. It is dominated by the factor

$$\frac{\mu_2}{r_0}\left(\frac{r}{r_0}\right)^2$$

or, neglecting periodic terms in f,

$$\frac{\mu_2 a^2 (1-e^2)^2}{a_0^3} = \frac{\mu_0^2 (1-e^2)^2}{2L^2}\left\{2\left(\frac{n_0}{n}\right)^2 \frac{1}{1+\frac{\mu_0}{\mu_2}}\right\}$$

Since μ_0/μ_2 is small ($\sim 1/80$), the relative order of magnitude of these terms is given by $2(n_0/n)^2$. For a high orbiter ($a \simeq 5\ R_{☾}$),

$$2\left(\frac{n_0}{n}\right)^2 \simeq 2 \times 10^{-3}.$$

Consider the sun's perturbation. The distance r_{03} is not well defined since the motion of the moon around the sun is far from a Keplerian motion. Instead of r_{03}, r_3 should be used. Then

$$r_{03}^2 = \left(\frac{m_2}{m_0+m_2} r_0\right)^2 + r_3^2 - 2\left(\frac{m_2}{m_0+m_2} r_0\right) r_3 \cos S_{03}$$

or

$$\frac{1}{r_{03}} = \frac{1}{r_3}\left\{1 + \left(\frac{m_2}{m_0+m_2}\ \frac{r_0}{r_3}\right)^2 - 2\,\frac{m_2}{m_0+m_2}\,\frac{r_0}{r_3} \cos S_{03}\right\}^{-1/2} =$$

$$= \frac{1}{r_3} \sum_{q=0}^{\infty} \left(\frac{m_2}{m_0 + m_2}\right)^q \left(\frac{r_0}{r_3}\right)^q P_q (\cos S_{03}).$$

The terms due to the sun's perturbation are therefore

$$\frac{\mu_3}{r_3} \sum_{q=0}^{\infty} \left(\frac{m_2}{m_0 + m_2}\right)^q \left(\frac{r_0}{r_3}\right)^q P_q (\cos S_{03}) \cdot$$

$$\cdot \sum_{p=2}^{\infty} \left(\frac{r}{r_3}\right)^p P_p (\cos S''_{13}) \left\{ \sum_{q=0}^{\infty} \left(\frac{m_2}{m_0 + m_2}\right)^q \left(\frac{r_0}{r_3}\right)^q P_q (\cos S_{03}) \right\}^p =$$

$$= \frac{\mu_3}{r_3} \left\{1 + \frac{m_2}{m_0 + m_2} \frac{r_0}{r_3} \cos S_{03} + \dots \right\} \cdot$$

$$\cdot \left\{ \left(\frac{r}{r_3}\right)^2 P_2 (\cos S''_{13}) \left[1 + \left(\frac{m_2}{m_0 + m_2}\right) \frac{r_0}{r_3} \cos S_{03} \right.\right.$$

$$\left. + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 P_2 (\cos S_{03}) + \dots \right]^2$$

$$+ \left(\frac{r}{r_3}\right)^3 P_3 (\cos S''_{13}) \left[1 + \left(\frac{m_2}{m_0 + m_2}\right) \frac{r_0}{r_3} \cos S_{03} \right.$$

$$\left.\left. + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 P_2 (\cos S_{03}) + \dots \right]^3 + \dots \right\} =$$

$$= \frac{\mu_3}{r_3} \left(\frac{r}{r_3}\right)^2 \left\{1 + \frac{m_2}{m_0 + m_2} \frac{r_0}{r_3} \cos S_{03} + \dots \right\} \cdot$$

$$\cdot \left\{ P_2(\cos S''_{13}) \left[1 + \left(\frac{m_2}{m_0 + m_2} \right) \frac{r_0}{r_3} \cos S_{03} + \cdots \right]^2 \right.$$

$$\left. + \left(\frac{r}{r_3} \right) P_3 (\cos S''_{13}) \left[1 + \left(\frac{m_2}{m_0 + m_2} \right) \frac{r_0}{r_3} \cos S_{03} + \cdots \right]^3 + \cdots \right\}$$

and they are dominated by the factor

$$\frac{\mu_3 \, a^2 \, (1 - e^2)^2}{a_3^3} = \frac{n_3^2 \, a^2 \, (1 - e^2)^2}{1 + \dfrac{\mu_0 + \mu_2}{\mu_3}} = \frac{\mu_0^2 \, (1 - e^2)^2}{2L^2} \left\{ 2 \left(\frac{n_3}{n} \right)^2 \frac{1}{1 + \dfrac{\mu_0 + \mu_2}{\mu_3}} \right\}.$$

The relative order of magnitude is given by

$$2 \left(\frac{n_3}{n} \right)^2 \simeq 2 \times 10^{-5}$$

for a high orbiter. Since, in the extreme case, $a/a_0 \simeq 3 \times 10^{-2}$, and if terms whose magnitude is less than 10^{-5} are excluded, the disturbing function due to the earth is reduced to

$$\frac{\mu_2}{r_0} \left(\frac{r}{r_0} \right)^2 \left\{ P_2 (\cos S'_{10}) + \frac{r}{r_0} P_3 (\cos S'_{10}) \right\}.$$

For the sun, the only term to be included is

$$\frac{\mu_3}{r_3} \left(\frac{r}{r_3} \right)^2 P_2 (\cos S''_{13})$$

since, in the extreme case, $a/a_3 \simeq 0.7 \times 10^{-4}$.

Therefore, including terms of magnitude 10^{-5}, the Hamiltonian becomes

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \left(\frac{r}{r_0}\right)^2 \left\{P_2 (\cos S'_{10}) + \frac{r}{r_0} P_3 (\cos S'_{10})\right\}$$

$$+ \frac{\mu_3}{r_3} \left(\frac{r}{r_3}\right)^2 P_2 (\cos S''_{13}) + \frac{\mu_0}{r} \left\{- \sum_{n=2}^{5} \left(\frac{R_c}{r}\right)^n J_n P_n (\sin \beta)\right.$$

$$\left. + \left(\frac{R_c}{r}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2 (\lambda' - \lambda_{22})\right\}.$$

It is assumed that J_3, J_4, and J_5 are of the third order. At this time it is worthwhile to point out that the inclusion of the indicated spherical harmonics in the potential field of the moon in the manner described above is an assumption, since this field is not very well known. It is possible that in the actual case one or more of J_3, J_4 and J_5 may be greater than J_2 or J_{22}. Our basic assumption is that the moon approximately satisfies the hydrostatic equilibrium conditions.

6. The Angles S'_{10} and S''_{13}

In view of the form assigned to the oblateness terms, the plane of reference is the equator of the moon. Therefore, the next step is to express the angles S'_{10} and S''_{13} in terms of the orbital elements of the orbiter, the moon, the earth and the sun, with respect to that plane. The geometry is shown in Figure 2.

The explicit form of S'_{10} and S''_{13} requires the solution of two spherical quadrangles. This is now done.

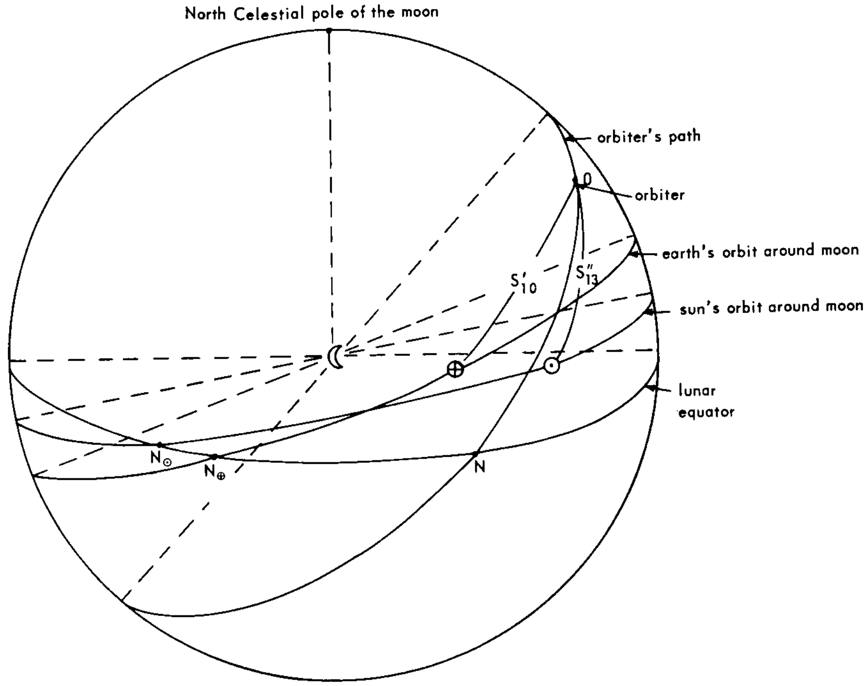

Figure 2

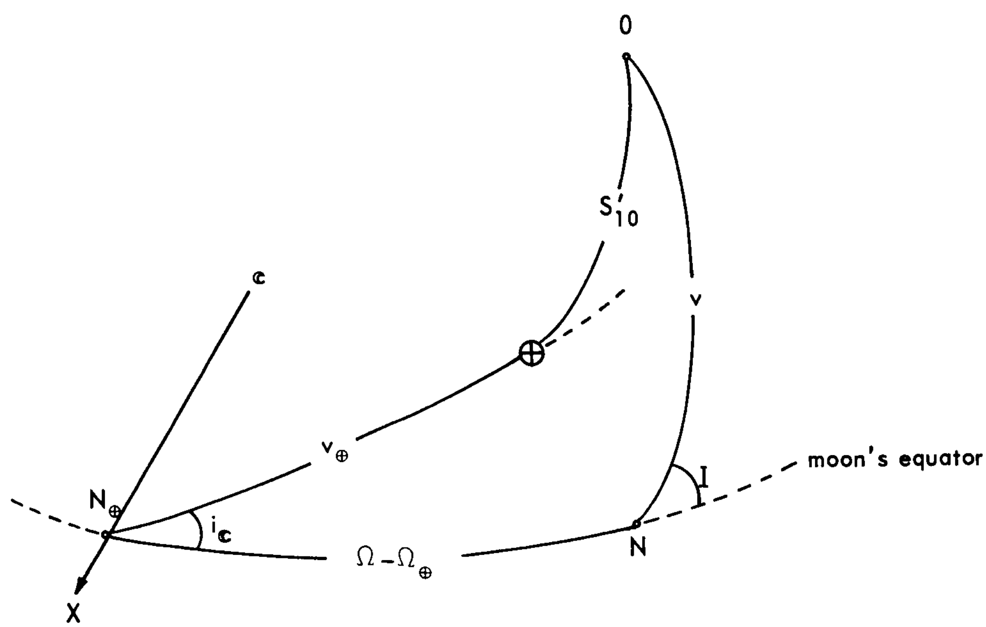

Figure 3

7. The Cosine of S'_{10}

If x, y, z and $x_\oplus$, $y_\oplus$, $z_\oplus$ are the rectangular coordinates of the orbiter and the earth, respectively, and r and $r_\oplus$ their selenocentric distances, then

$$\cos S'_{10} = \frac{\mathbf{r} \cdot \mathbf{r}_\oplus}{r\, r_\oplus} .$$

Consider

$$\Delta\Omega = \Omega - \Omega_\oplus$$

$$v = f + \omega$$

$$v_\oplus = f_\oplus + \omega_\oplus .$$

Then, using vector notation, in the equatorial (moon) system indicated in Figure 3,

$$\mathbf{r}_\oplus = r_\oplus \begin{pmatrix} \cos v_\oplus \\ \sin v_\oplus \cos i_☾ \\ \sin v_\oplus \sin i_☾ \end{pmatrix}$$

$$\mathbf{r} = r \begin{pmatrix} (\cos\omega \cos\Delta\Omega - \sin\omega \sin\Delta\Omega \cos I) \cos f \\ \qquad - (\sin\omega \cos\Delta\Omega + \cos\omega \sin\Delta\Omega \cos I) \sin f \\ (\cos\omega \sin\Delta\Omega + \sin\omega \cos\Delta\Omega \cos I) \cos f \\ \qquad - (\sin\omega \sin\Delta\Omega - \cos\omega \cos\Delta\Omega \cos I) \sin f \\ \sin\omega \sin I \cos f + \cos\omega \sin I \sin f \end{pmatrix}$$

This enables us to compute $\cos S'_{10}$.

8. The Cosine of S''_{13}

In exactly the same way the $\cos S''_{13}$ is obtained by substituting ☉ for ⊕ in the formulas in section 7.

9. The Main Problem

The following approximations can be introduced:

a. The earth's orbit around the moon (or vice versa) lies on the lunar equatorial plane. The error introduced by this approximation is proportional to the sine of half the inclination of the lunar orbit to its equator ($\sim 6° 41'$), or about 0.06.

b. The orbit of the moon around the earth is circular and the motion uniform. The error is proportional to the moon's eccentricity or about 0.055.

c. The sun's perturbations are negligible. The relative magnitude for a moderately high satellite is about 0.05.

d. The mean longitude of the earth, $\lambda_\oplus$, is equal to λ_{22}.

If these approximations are introduced, the precision of the disturbing function is not higher than 10^{-4}. It will be called the "disturbing function of the main problem," and it is given by

$$\tilde{\tilde{F}} = \frac{\mu_0^2}{2L^2} + \frac{n_0^2}{\epsilon} r^2 P_2 (\cos \tilde{S}'_{10})$$

$$+ \frac{\mu_0}{r}\left(\frac{R_☾}{r}\right)^2 \left\{- J_2 P_2(\sin \beta) + J_{22} P_{22}(\sin \beta) \cos 2(\lambda' - \lambda_\oplus)\right\},$$

where

$$\epsilon = 1 + \frac{\mu_0}{\mu_2}.$$

In this case the $\cos S'_{10}$ is much simplified. Since $i_☾ = 0$ and $v_\oplus + \Omega_\oplus$ can be replaced by $\lambda_\oplus$, it then follows

$$\cos \tilde{S}'_{10} = \cos v \cos (\Omega - \lambda_\oplus) - \sin v \sin (\Omega - \lambda_\oplus) \cos I = s.$$

The equations of motion are given by the canonical set

$$\dot{L} = \frac{\partial \tilde{\tilde{F}}}{\partial l}, \qquad \dot{l} = -\frac{\partial \tilde{\tilde{F}}}{\partial L},$$

$$\dot{G} = \frac{\partial \tilde{\tilde{F}}}{\partial \omega}, \qquad \dot{\omega} = -\frac{\partial \tilde{\tilde{F}}}{\partial G},$$

$$\dot{H} = \frac{\partial \tilde{\tilde{F}}}{\partial \Omega}, \qquad \dot{\Omega} = -\frac{\partial \tilde{\tilde{F}}}{\partial H}.$$

Since the variables Ω and $\lambda_{\oplus}$ appear only in the combination $\Omega - \lambda_{\oplus}$, where $\lambda_{\oplus} = n_{\mathbb{C}}^* t + \text{const.}$ (approximation (b)), the degree of freedom is reduced by one by choosing as a new variable

$$h = \Omega - \lambda_{\oplus}.$$

The Hamiltonian must be modified accordingly, for

$$\dot{h} = \dot{\Omega} - n_{\mathbb{C}}^* = -\frac{\partial}{\partial H}(\tilde{\tilde{F}} + n_{\mathbb{C}}^* H)$$

(i.e., by using $\tilde{\tilde{F}} + n_{\mathbb{C}}^* H$ in place of $\tilde{\tilde{F}}$).

The Hamiltonian is still time dependent through $\lambda_{\oplus} = n_{\mathbb{C}}^* t + \text{const.}$ Since the longest meridian is always pointing toward the earth, it is possible to choose the rotating system whose x-axis passes through this meridian. The final form of the Hamiltonian for the main problem is therefore

$$F = \frac{\mu_0^2}{2L^2} + n_{\mathbb{C}}^* H + \frac{n_{\mathbb{C}}^2 r^2}{\epsilon} P_2(s)$$

$$+ \frac{\mu_0}{r}\left(\frac{R_{\mathbb{C}}}{r}\right)^2 \left\{- J_2 P_2(\sin\beta) + J_{22} P_{22}(\sin\beta)\cos 2\lambda\right\},$$

where

$$\lambda = \lambda' - \lambda_{\oplus}.$$

The equations of motion are (using g in place of ω):

$$\dot{L} = \frac{\partial F}{\partial l}, \quad \dot{G} = \frac{\partial F}{\partial g}, \quad \dot{H} = \frac{\partial F}{\partial h},$$

$$\dot{l} = -\frac{\partial F}{\partial L}, \quad \dot{g} = -\frac{\partial F}{\partial G}, \quad \dot{h} = -\frac{\partial F}{\partial H}.$$

In the discussion that follows the solution of this system will be given using the already well known von Zeipel's method.

10. Development of the Disturbing Function

The notation used will be as shown in Figure 4.

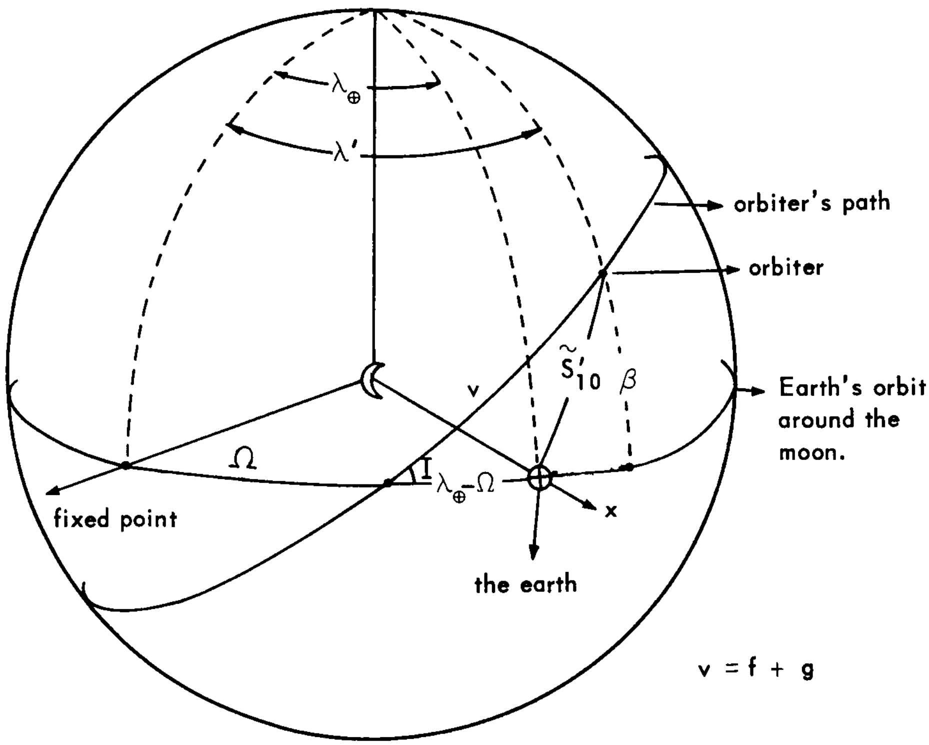

Figure 4

$$\cos \tilde{S}'_{10} = s = \cos (v + \Omega - \lambda_{\oplus}) - 2 \sin^2 \frac{I}{2} \sin v \sin (\lambda_{\oplus} - \Omega)$$

and

$$P_2(s) = \frac{3}{2} s^2 - \frac{1}{2} =$$

$$= \frac{3}{2} \Big\{ \cos^2 f \cos (2g + 2h) + \sin^2 (g + h)$$

$$- \sin f \cos f \sin (2g + 2h) + 4 \sin^4 \frac{I}{2} \sin^2 h [\cos^2 g - \cos^2 f \cos 2g$$

$$+ \sin f \cos f \sin 2g] + 4 \sin^2 \frac{I}{2} \sin h [\sin f \cos f \cos (2g + h)$$

$$- \cos g \sin (g + h) + \cos^2 f \sin (2g + h)] \Big\} - 1/2 .$$

Further,

$$\sin \beta = \sin I \sin v = \sin I \sin (f + g)$$

$$\cos \beta \cos (\lambda - h) = \cos v$$

$$\cos \beta \sin (\lambda - h) = \cos I \sin v ,$$

and therefore

$$\cos \beta \cos \lambda = \cos v \cos h - \cos I \sin v \sin h ,$$

from which

$$P_2(\sin \beta) = \frac{3}{2} \sin^2 \beta - \frac{1}{2} = \frac{3}{2} \sin^2 I \sin^2 (f + g) - \frac{1}{2}$$

$$P_{22}(\sin \beta) \cos 2\lambda = 6 \cos^2 \beta \cos^2 \lambda - 3 \cos^2 \beta =$$

$$6(\zeta^2 \cos^2 f + \chi^2 \sin^2 f + 2\zeta\chi \sin f \cos f) - 3(1 - \sin^2 I \sin^2 v),$$

where

$$\zeta = \cos g \cos h - \cos I \sin g \sin h$$

$$\chi = -\sin g \cos h - \cos I \cos g \sin h .$$

Therefore, writing $\mu_{\mathfrak{C}}$ for μ_0, we have

$$F = \frac{\mu_{\mathfrak{C}}^2}{2L^2} + n_{\mathfrak{C}}^* H + \frac{3}{2}\frac{n_{\mathfrak{C}}^2 r^2}{\epsilon}\Big\{\cos^2 f \cos(2g + 2h)$$

$$+ \sin^2(g + h) - \sin f \cos f \sin(2g + 2h) + 4\sin^4\frac{I}{2}\sin^2 h\,[\cos^2 g$$

$$- \cos^2 f \cos 2g + \sin f \cos f \sin 2g] + 4\sin^2\frac{I}{2}\sin h\,[\sin f \cos f \cdot$$

$$\cdot \cos(2g + h) - \cos g \sin(g + h) + \cos^2 f \sin(2g + h)]\Big\}$$

$$-\frac{1}{2\epsilon} n_{\mathfrak{C}}^2 r^2 + \frac{\mu_{\mathfrak{C}} R_{\mathfrak{C}}^2}{r^3}\Big\{-\frac{3}{2} J_2 \sin^2 I \sin^2(f + g)$$

$$+\frac{1}{2} J_2 + J_{22}[6(\zeta^2 \cos^2 f + \chi^2 \sin^2 f + 2\zeta\chi \sin f \cos f) - 3(1 - \sin^2 I \sin^2 v)]\Big\}$$

$$= F_0 + F_1 + F_2 ,$$

where

$$F_0 = \frac{\mu_{\mathfrak{C}}^2}{2L^2} \qquad \text{(0th order)}$$

$$F_1 = n_{\mathfrak{C}}^* H \qquad \text{(1st order)}$$

$$F_2 = F - (F_1 + F_0) \qquad \text{(2nd order)}.$$

11. Elimination of Short Periodic Perturbations

The terms in F which depend on l correspond to the short periodic perturbations. According to von Zeipel's method, the elimination of these terms corresponds to the solution of the system

$$F_0'(L') = F_0(L')$$

$$F_1'(H') = F_1(H') + \frac{\partial S_1}{\partial l} \frac{\partial F_0}{\partial L'}$$

$$F_2' \qquad = F_2 + \frac{\partial S_1}{\partial h} \frac{\partial F_1}{\partial H'} + \frac{\partial S_2}{\partial l} \frac{\partial F_0}{\partial L'}$$

$$+ \frac{1}{2} \left(\frac{\partial S_1}{\partial l}\right)^2 \frac{\partial^2 F_0}{\partial L'^2}$$

$$\cdots\cdots\cdots\cdots\cdot \quad ,$$

where the generating function of the transformation

$$(L, G, H, l, g, h) \rightarrow (L', G', H', l', g', h')$$

is

$$S = L'l + G'g + H'h + S_1 + S_2 + \cdots ,$$

and the new Hamiltonian

$$F' = F_0' + F_1' + F_2' + \cdots$$

should be independent of l'.

In order to accomplish this last requirement, a particular solution is given by

$$S_1 = 0$$

$$F_2' = F_{2s}$$

$$\frac{\partial S_2}{\partial l} \frac{\partial F_0}{\partial L'} = - F_{2p} ,$$

where F_{2s} and F_{2p} are, respectively, the parts of F_2 independent and dependent on $\underline{l}$.

If S_2 is determined from this system, then the relations between old and new variables are

$$l' = l + \frac{\partial S_2}{\partial L'} , \quad g' = g + \frac{\partial S_2}{\partial G'} , \quad h' = h + \frac{\partial S_2}{\partial H'} ,$$

$$L = L' + \frac{\partial S_2}{\partial l} , \quad G = G' + \frac{\partial S_2}{\partial g} , \quad H = H' + \frac{\partial S_2}{\partial h} .$$

Next, the "secular" part of F_2 is determined. By definition,

$$F_{2s} = \frac{1}{2\pi} \int_0^{2\pi} F_2 \, dl .$$

The integrals needed are

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \cos^2 f \, dl = (1 + 4\, e^2)\, \frac{a^2}{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \, dl = \left(1 + \frac{3}{2}\, e^2\right) a^2$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \sin f \cos f \, dl = 0\,,$$

which are evaluated using the relations (E = eccentric anomaly)

$$dl = (1 - e \cos E)\, dE$$

$$r \cos f = a\,(\cos E - e)$$

$$r \sin f = a\,\sqrt{1 - e^2}\, \sin E$$

$$r = a\,(1 - e \cos E).$$

Furthermore, the following integrals are also needed:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^3} \cos 2f \, dl = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^3}\, dl = \frac{1}{a^3 (1 - e^2)^{3/2}}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^3} \sin f \cos f \, dl = 0.$$

These are evaluated using the relations

$$dl = \frac{r^2}{a^2 (1 - e^2)^{1/2}}\, df$$

$$\frac{1}{r} = \frac{1 + e \cos f}{a (1 - e^2)}.$$

If the preceding results are used, it follows that

$$F' = \frac{\mu_{\mathbb{C}}^2}{2L'^2} + n_{\mathbb{C}}^* H' + \frac{n_{\mathbb{C}}^2 L'^4}{16 \mu_{\mathbb{C}}^2 \epsilon} \left\{ \left(5 - 3\frac{G'^2}{L'^2}\right)\left[\left(-1 + 3\frac{H'^2}{G'^2}\right)\right.\right.$$

$$\left. + 3\left(1 - \frac{H'^2}{G'^2}\right) \cos 2h'\right] + 15\left(1 - \frac{G'^2}{L'^2}\right)\left[\frac{1}{2}\left(1 + \frac{H'}{G'}\right)^2 \cos 2(g' + h')\right.$$

$$\left.\left. + \left(1 - \frac{H'^2}{G'^2}\right) \cos 2g' + \frac{1}{2}\left(1 - \frac{H'}{G'}\right)^2 \cos 2(g' - h')\right]\right\}$$

$$+ \frac{1}{4}\frac{\mu_{\mathbb{C}}^4}{L'^6}\left(\frac{L'}{G'}\right)^3 \left[- R_{\mathbb{C}}^2 J_2 \left(1 - 3\frac{H'^2}{G'^2}\right) + 6R_{\mathbb{C}}^2 J_{22} \left(1 - \frac{H'^2}{G'^2}\right) \cos 2h'\right],$$

and since

$$\dot{L}' = \frac{\partial F'}{\partial l'} = 0,$$

then

$$L' = \text{const.}$$

The generating function and the short period perturbations are obtained next. Consider F in the form

$$F = \frac{\mu_{\mathbb{C}}^2}{2L^2} + n_{\mathbb{C}}^* H + \frac{\mu_{\mathbb{C}} \bar{R}_{\mathbb{C}}^2}{r^3} \left[- J_2 P_2 (\sin \beta) + J_{22} P_{22} (\sin \beta) \quad \cos 2\lambda\right]$$

$$+ \frac{n_{\mathbb{C}}^2 a_{\mathbb{C}}^3}{\epsilon} \frac{1}{a_{\mathbb{C}}} \left(\frac{r}{a_{\mathbb{C}}}\right)^2 P_2(s),$$

and write

$$U_1 = r^2 P_2(s)$$

$$U_{1s} = \frac{1}{2\pi} \int_0^{2\pi} r^2 P_2(s) \, dl$$

$$U_{1p} = r^2 P_2(s) - U_{1S}$$

$$U_2 = \frac{1}{r^3} P_{22} (\sin \beta) \cos 2\lambda$$

$$U_{2S} = \frac{1}{2\pi} \int_0^{2\pi} U_2 \, dl$$

$$U_{2p} = U_2 - U_{2S}$$

$$U_3 = -\frac{1}{r^3} P_2 (\sin \beta)$$

$$U_{3S} = \frac{1}{2\pi} \int_0^{2\pi} U_3 \, dl$$

$$U_{3p} = U_3 - U_{3S}.$$

Then

$$F_{2p} = \mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_2 U_{3p} + \mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_{22} U_{2p} + \frac{n_{\mathbb{C}}^2}{\epsilon} U_{1p}.$$

Further,

$$\frac{\partial F_0}{\partial L'} = -\frac{\mu_{\mathbb{C}}^2}{L'^3} = -n',$$

so that the differential equation for S_2 is

$$n' \frac{\partial S_2}{\partial l} = F_{2p},$$

or

$$S_2 = \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_2}{n'} \int U_{3p} \, dl + \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_{22}}{n'} \int U_{2p} \, dl$$

$$+ \frac{n_{\mathbb{C}}^2}{\epsilon n'} \int U_{1p} \, dl.$$

The evaluation of the integrals is carried out using E or f as independent variables.

It is found that

$$\int U_{3p}\, dl = -D\left(\frac{3}{4}\sin^2 I' - \frac{1}{2}\right) + \frac{3}{4}K\sin^2 I'$$

$$+ \frac{(1-e'^2)^{-3/2}}{a'^3}\left(\frac{3}{4}\sin^2 I' - \frac{1}{2}\right) l$$

$$\int U_{2p}\, dl = (6\chi'^2 - 3 + 3\sin^2 I' \cos^2 g)\, D$$

$$+ \left[6(\zeta'^2 - \chi'^2) - 3\sin^2 I' \cos 2g\right] Q$$

$$+ \left[12\zeta'\chi' + 3\sin^2 I' \sin 2g\right] J$$

$$- \frac{3l}{2a'^3}(1-e'^2)^{-3/2}\sin^2 I' \cos 2h$$

$$\int U_{1p}\, dl = A\left\{\frac{3}{2}\cos(2g+2h) - 6\sin^4\frac{I'}{2}\sin^2 h\cos 2g\right.$$

$$\left. + 6\sin^2\frac{I'}{2}\sin h\sin(2g+h)\right\}$$

$$+ B\left\{\frac{3}{2}\sin^2(g+h) + 6\sin^4\frac{I'}{2}\sin^2 h\cos^2 g\right.$$

$$- 6 \sin^2 \frac{I'}{2} \sin h \cos g \sin (g + h) - \frac{1}{2} \Big\}$$

$$+ C \Big\{ - \frac{3}{2} \sin (2g + 2h) + 6 \sin^4 \frac{I'}{2} \sin^2 h \sin 2g$$

$$+ 6 \sin^2 \frac{I'}{2} \sin h \cos (2g + h) \Big\}$$

$$- \frac{a'^2 \, l}{16} \Big\{ (2 + 3 e'^2) \quad \left[(-1 + 3 \cos^2 I') + 3 \sin^2 I' \cos 2h \right]$$

$$+ 15 e'^2 \Big[\frac{1}{2} (1 + \cos I')^2 \cos (2g + 2h) + \sin^2 I' \cos 2g$$

$$+ \frac{1}{2} (1 - \cos I')^2 \cos (2g - 2h) \Big] \Big\}$$

where

$$a' = \frac{L'^2}{\mu_{☾}}$$

$$e' = \left(1 - \frac{G'^2}{L'^2} \right)^{1/2}$$

$$\cos I' = \frac{H'}{G'}$$

$$\zeta' = \cos g \cos h - \cos I' \sin g \sin h$$

$$\chi' = - \sin g \cos h - \cos I' \cos g \sin h$$

$$A = \int r'^2 \cos^2 f' \, dl = a'^2 \left[\left(\frac{1}{2} + 2 e'^2 \right) E' - \left(\frac{33}{12} e' + e'^3 \right) \sin E' \right.$$

$$\left. + \left(\frac{1}{4} + \frac{e'^2}{2} \right) \sin 2 E' - \frac{e'}{12} \sin 3 E' \right]$$

$$B = \int r'^2 \, dl = a'^2 \left[\left(1 + \frac{3}{2} e'^2 \right) E' - 3 e' \left(1 + \frac{1}{4} e'^2 \right) \sin E' \right.$$

$$\left. + \frac{3}{4} e'^2 \sin 2 E' - \frac{e'^3}{12} \sin 3 E' \right]$$

$$C = \int r'^2 \sin f' \cos f' \, dl = a'^2 \frac{G'}{L'} \left[\frac{5}{4} e' \cos E' - \frac{1}{4} (1 + e'^2) \cos 2 E' \right.$$

$$\left. + \frac{e'}{12} \cos 3 E' \right]$$

$$D = \int \frac{1}{r'^3} \, dl = a'^{-3} (1 - e'^2)^{-3/2} \left[f' + e' \sin f' \right]$$

$$Q = \int \frac{\cos^2 f'}{r'^3}\, dl = a'^{-3} (1 - e'^2)^{-3/2} \left[\frac{1}{2} f' + \frac{3e'}{4} \sin f' \right.$$

$$\left. + \frac{1}{4} \sin 2 f' + \frac{e'}{12} \sin 3 f' \right]$$

$$J = \int \frac{\sin f' \cos f'}{r'^3}\, dl = a'^{-3} (1 - e'^2)^{-3/2} \left[- \frac{e'}{4} \cos f' - \frac{1}{4} \cos 2 f' \right.$$

$$\left. - \frac{e'}{12} \cos 3 f' \right]$$

$$K = \int \frac{\cos 2(f' + g)}{r'^3}\, dl = a'^{-3} (1 - e'^2)^{-3/2} \left[\frac{1}{2} \sin (2g + 2f') \right.$$

$$\left. + \frac{e'}{2} \sin (2g + f') + \frac{e'}{6} \sin (2g + 3 f') \right]$$

and

$$E' - e' \sin E' = l \qquad \text{(defines } E'\text{)}$$

$$r' = a' (1 - e' \cos E') \qquad \text{(defines } r'\text{)}$$

$$\frac{1}{r'} = \frac{1 + e' \cos f'}{a' (1 - e'^2)} \qquad \text{(defines } f'\text{).}$$

In the integrals above, any constant with respect to the variable of integration (E' or f') has been neglected. At this stage, S_2 is completely defined. After some simplification, the final result is

$$S_2 = \frac{1}{8}\frac{\mu_{☾} R_{☾}^2 J_2}{n' a'^3 (1-e'^2)^{3/2}} \Big\{-2(1-3\cos^2 I')\left[e' \sin f' + (f' - l)\right] + \sin^2 I' \left[3e' \sin(f' + 2g)\right.$$

$$\left.+ 3\sin(2f' + 2g) + e' \sin(3f' + 2g)\right]\Big\}$$

$$+ \frac{1}{8}\frac{\mu_{☾} R_{☾}^2 J_{22}}{n' a'^3 (1-e'^2)^{3/2}} \Big\{6\sin^2 I' \left[2(f' - l)\cos 2h + e' \sin(f' - 2h) + e' \sin(f' + 2h)\right]$$

$$+ (1-\cos I')^2 \left[3e' \sin(f' + 2g - 2h) + 3\sin(2f' + 2g - 2h) + e' \sin(3f' + 2g - 2h)\right]$$

$$+ (1+\cos I')^2 \left[3e' \sin(f' + 2g + 2h) + 3\sin(2f' + 2g + 2h) + e' \sin(3f' + 2g + 2h)\right]\Big\}$$

$$+ \frac{1}{384}\frac{n_{☾}^2 a'^2}{\epsilon n'} \Big\{12\left[-2(1-3\cos^2 I')(2+3e'^2) + 6\sin^2 I'(2+3e'^2)\cos 2h\right.$$

$$+ 30e'^2 \sin^2 I' \cos 2g$$

$$\left.+ 15e'^2(1-\cos I')^2 \cos(2g-2h) + 15e'^2(1+\cos I')^2\cos(2g+2h)\right] \cdot (E' - l)$$

$$+ 4(1-3\cos^2 I')\left[9e'(4+e'^2)\sin E' - 9e'^2 \sin 2E' + e'^3 \sin 3E'\right]$$

$$+ 6\sin^2 I' \left[-9e'(4+e'^2)\sin(E' - 2h) + 9e'^2\sin(2E' - 2h) - e'^3 \sin(3E' - 2h)\right.$$

$$- 9e'(4+e'^2)\sin(E' + 2h) + 9e'^2 \sin(2E' + 2h) - e'^3 \sin(3E' + 2h)$$

$$- 15e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\sin(E' - 2g)$$

$$+ 3\{(2+e'^2) - 2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E' - 2g)$$

$$- e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\sin(3E' - 2g)$$

$$- 15e'\{(2+e'^2) + 2(1-e'^2)^{1/2}\}\sin(E' + 2g)$$

$$+ 3\{(2+e'^2) + 2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E' + 2g)$$

$$\left.- e'\{(2-e'^2) + 2(1-e'^2)^{1/2}\}\sin(3E' + 2g)\right]$$

$$+ 3(1-\cos I')^2\left[-15e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\sin(E' - 2g + 2h)\right.$$

$$+ 3\{(2+e'^2) - 2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E' - 2g + 2h)$$

$$- e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\sin(3E' - 2g + 2h)$$

$$- 15 e' \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \sin (E' + 2g - 2h)$$

$$+ 3 \{(2 + e'^2) + 2(1 - e'^2)^{1/2} (1 + e'^2)\} \sin (2E' + 2g - 2h)$$

$$- e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \sin (3E' + 2g - 2h)\Big]$$

$$+ 3 (1 + \cos I')^2 \Big[- 15 e' \{(2 + e'^2) - 2(1 - e'^2)^{1/2}\} \sin (E' - 2g - 2h)$$

$$+ 3 \{(2 + e'^2) - 2(1 - e'^2)^{1/2} (1 + e'^2)\} \sin(2E' - 2g - 2h)$$

$$- e' \{(2 - e'^2) - 2(1 - e'^2)^{1/2}\} \sin (3E' - 2g - 2h)$$

$$- 15 e' \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \sin(E' + 2g + 2h)$$

$$+ 3 \{(2 + e'^2) + 2(1 - e'^2)^{1/2} (1 + e'^2)\} \sin(2E' + 2g + 2h)$$

$$- e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \sin (3E' + 2g + 2h)\Big]\Big\},$$

so that, for the short periodic terms,

$$S = L' l + G' g + H' h + S_2$$

to second order.

The short period perturbations are given by

$$l' = \frac{\partial S}{\partial L'} = l + \frac{\partial S_2}{\partial L'}$$

$$g' = \frac{\partial S}{\partial G'} = g + \frac{\partial S_2}{\partial G'}$$

$$h' = \frac{\partial S}{\partial H'} = h + \frac{\partial S_2}{\partial H'}$$

$$L = \frac{\partial S}{\partial l} = L' + \frac{\partial S_2}{\partial l}$$

$$G = \frac{\partial S}{\partial g} = G' + \frac{\partial S_2}{\partial g}$$

$$H = \frac{\partial S}{\partial h} = H' + \frac{\partial S_2}{\partial h}.$$

It is easier to compute the partial derivatives with respect to the Keplerian elements a', e', I' and then compute them with respect to L', G', H' as follows:

$$\frac{\partial S_2}{\partial L'} = \frac{\partial S_2}{\partial a'} \frac{\partial a'}{\partial L'} + \frac{\partial S_2}{\partial e'} \frac{\partial e'}{\partial L'} = 2 \frac{L'}{\mu_{\mathbb{C}}} \frac{\partial S_2}{\partial a'} + \frac{G'^2}{e' L'^3} \frac{\partial S_2}{\partial e'}$$

$$\frac{\partial S_2}{\partial G'} = \frac{\partial S_2}{\partial e'} \frac{\partial e'}{\partial G'} + \frac{\partial S_2}{\partial I'} \frac{\partial I'}{\partial G'} = - \frac{G'}{e' L'^2} \frac{\partial S_2}{\partial e'} + \frac{H'}{G'^2 \sin I'} \frac{\partial S_2}{\partial I'}$$

$$\frac{\partial S_2}{\partial H'} = \frac{\partial S_2}{\partial I'} \frac{\partial I'}{\partial H'} = - \frac{1}{G' \sin I'} \frac{\partial S_2}{\partial I'}.$$

Furthermore, it is important to note that

$$\frac{\partial f'}{\partial e'} = \left(\frac{a'}{r'} + \frac{L'^2}{G'^2} \right) \sin f'$$

$$\frac{\partial E'}{\partial e'} = \frac{a'}{r'} \sin E'$$

$$\frac{\partial n'}{\partial a'} = - \frac{3}{2} \frac{n'}{a'} \quad .$$

The partial derivative with respect to l does not need to be computed, since

$$\frac{\partial S_2}{\partial l} = \frac{1}{n'} F_{2p} = \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_2}{n'} U_{3p} + \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_{22}}{n'} U_{2p} + \frac{n_{\mathbb{C}}^2}{\epsilon n'} U_{1p} = \frac{1}{n'}(F - F') .$$

To simplify the formulae, the following notations are now introduced:

$$B_{2,1} = e' \sin f' + (f' - l)$$

$$B_{2,2} = 3\, e' \sin(f' + 2g) + 3 \sin(2f' + 2g) + e' \sin(3f' + 2g)$$

$$B_{22,1} = 2(f' - l) \cos 2h + e' \sin(f' - 2h) + e' \sin(f' + 2h)$$

$$B_{22,2} = 3\, e' \sin(f' + 2g - 2h) + 3 \sin(2f' + 2g - 2h) + e' \sin(3f' + 2g - 2h)$$

$$B_{22,3} = 3\, e' \sin(f' + 2g + 2h) + 3 \sin(2f' + 2g + 2h) + e' \sin(3f' + 2g + 2h)$$

$$B_{\mathbb{C},1} = 9\, e' (4 + e'^2) \sin E' - 9\, e'^2 \sin 2E' + e'^3 \sin 3E'$$

$$\begin{aligned}
B_{\mathbb{C},2} = & - 9\, e' (4 + e'^2) \sin(E' - 2h) + 9\, e'^2 \sin(2E' - 2h) - e'^3 \sin(3E' - 2h) \\
& - 9\, e' (4 + e'^2) \sin(E' + 2h) + 9\, e'^2 \sin(2E' + 2h) - e'^3 \sin(3E' + 2h) \\
& - 15\, e' \{(2 + e'^2) - 2(1 - e'^2)^{1/2}\} \sin(E' - 2g) + 3\{(2 + e'^2) - 2(1 - e'^2)^{1/2} (1 + e'^2)\} \cdot \\
& \cdot \sin(2E' - 2g) - e' \{(2 - e'^2) - 2(1 - e'^2)^{1/2}\} \sin(3E' - 2g) - 15\, e' \{(2 + e'^2) \\
& + 2 (1 - e'^2)^{1/2}\} \sin(E' + 2g) + 3 \{(2 + e'^2) + 2 (1 - e'^2)^{1/2} (1 + e'^2)\} \sin(2E' + 2g) \\
& - e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \sin(3E' + 2g)
\end{aligned}$$

$$B_{\mathbb{C},3} = -15\,e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\sin(E'-2g+2h) + 3\{(2+e'^2)$$

$$-2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E'-2g+2h) - e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\cdot$$

$$\cdot\sin(3E'-2g+2h) - 15\,e'\{(2+e'^2) + 2(1-e'^2)^{1/2}\}\sin(E'+2g-2h)$$

$$+3\{(2+e'^2) + 2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E'+2g-2h) - e'\{(2-e'^2)$$

$$+2(1-e'^2)^{1/2}\}\sin(3E'+2g-2h)$$

$$B_{\mathbb{C},4} = -15\,e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\sin(E'-2g-2h) + 3\{(2+e'^2)$$

$$-2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E'-2g-2h) - e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\cdot$$

$$\cdot\sin(3E'-2g-2h) - 15\,e'\{(2+e'^2) + 2(1-e'^2)^{1/2}\}\sin(E'+2g+2h)$$

$$+3\{(2+e'^2) + 2(1-e'^2)^{1/2}(1+e'^2)\}\sin(2E'+2g+2h)$$

$$-e'\{(2-e'^2) + 2(1-e'^2)^{1/2}\}\sin(3E'+2g+2h)\,.$$

With this notation, it follows that

$$\frac{\partial S_2}{\partial a'} = -\frac{3}{16}\,\frac{n' R_{\mathbb{C}}^2 J_2}{a'(1-e'^2)^{3/2}}\,\{-2(1-3\cos^2 I')\,B_{2,1} + (\sin^2 I')\,B_{2,2}\}$$

$$-\frac{3}{16}\,\frac{n' R_{\mathbb{C}}^2 J_{22}}{a'(1-e'^2)^{3/2}}\,\{6(\sin^2 I')B_{22,1} + (1-\cos I')^2 B_{22,2} + (1+\cos I')^2 B_{22,3}\}$$

$$+\frac{7}{768}\,\frac{n_{\mathbb{C}}^2 a'}{\epsilon n'}\,\{12\,[-2(1-3\cos^2 I')(2+3e'^2) + 6\sin^2 I'(2+3e'^2)\cos 2h$$

$$+30\,e'^2\sin^2 I'\cos 2g + 15\,e'^2(1-\cos I')^2\cos(2g-2h) + 15\,e'^2(1+\cos I')^2\cdot$$

$$\cdot\cos(2g+2h)]\cdot(E'-l) + 4(1-3\cos^2 I')\,B_{\mathbb{C},1} + 6(\sin^2 I')\,B_{\mathbb{C},2}$$

$$+3(1-\cos I')^2 B_{\mathbb{C},3} + 3(1+\cos I')^2 B_{\mathbb{C},4}\}$$

$$\frac{\partial S_2}{\partial I'} = \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_2}{(1-e'^2)^{3/2}} \sin I' \cos I' \{-6 B_{2,1} + B_{2,2}\}$$

$$+ \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_{22}}{(1-e'^2)^{3/2}} \sin I' \{6 (\cos I') B_{22,1} + (1-\cos I') B_{22,2} - (1+\cos I') B_{22,3}\}$$

$$+ \frac{1}{64} \frac{n' a'^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \sin I' \{12 [-2 \cos I' (2+3e'^2) + 2 \cos I' (2+3e'^2) \cos 2h$$

$$+ 10 e'^2 \cos I' \cos 2g + 5 e'^2 (1-\cos I') \cos (2g-2h) - 5e'^2 (1+\cos I') \cos(2g+2h)] .$$

$$\cdot (E' - l) + 4 (\cos I') B_{\mathbb{C},1} + 2 (\cos I') B_{\mathbb{C},2} + (1-\cos I') B_{\mathbb{C},3} - (1+\cos I') B_{\mathbb{C},4}\}$$

$$\frac{\partial S_2}{\partial e'} = \frac{3}{8} \frac{e' n' R_{\mathbb{C}}^2 J_2}{(1-e'^2)^{5/2}} \{-2 (1-3\cos^2 I') B_{2,1} + (\sin^2 I') B_{2,2}\}$$

$$+ \frac{1}{8} \frac{n' R_{\mathbb{C}}^2 J_2}{(1-e'^2)^{3/2}} \{-2 (1-3\cos^2 I') \sin f + \sin^2 I' [3 \sin(f'+2g) + \sin(3f'+2g)]\}$$

$$+ \frac{1}{8} \frac{n' R_{\mathbb{C}}^2 J_2}{(1-e'^2)^{3/2}} \{-2(1-3\cos^2 I') [e \cos f' + 1] + 3 \sin^2 I' [e' \cos(f'+2g)$$

$$+ 2 \cos(2f'+2g) + e' \cos(3f'+2g)]\} \left(\frac{a'}{r'} + \frac{L'^2}{G'^2}\right) \sin f$$

$$+ \frac{3}{8} \frac{e' n' R_{\mathbb{C}}^2 J_{22}}{(1-e'^2)^{5/2}} \{6(\sin^2 I') B_{22,1} + (1-\cos I')^2 B_{22,2} + (1+\cos I')^2 B_{22,3}\}$$

$$+\frac{1}{8}\frac{n' R_{\mathbb{C}}^2 J_{22}}{(1-e'^2)^{3/2}}\{6\sin^2 I'\,[\sin(f'-2h)+\sin(f'+2h)]+(1-\cos I')^2\,[3\sin(f'+2g-2h)$$

$$+\sin(3f'+2g-2h)]+(1+\cos I')^2\,[3\sin(f'+2g+2h)+\sin(3f'+2g+2h)]\}$$

$$+\frac{3}{8}\frac{n' R_{\mathbb{C}}^2 J_{22}}{(1-e'^2)^{3/2}}\{2\sin^2 I'\,[2\cos 2h+e'\cos(f'-2h)+e'\cos(f'+2h)]$$

$$+(1-\cos I')^2\,[e'\cos(f'+2g-2h)+2\cos(2f'+2g-2h)+e'\cos(3f'+2g-2h)]$$

$$+(1+\cos I')^2\,[e'\cos(f'+2g+2h)+2\cos(2f'+2g+2h)+e'\cos(3f'+2g+2h)]\}\cdot$$

$$\cdot\left(\frac{a'}{r'},+\frac{L'^2}{G'^2}\right)\sin f'$$

$$+\frac{1}{128}\frac{n'a^2}{\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\Big\{24\,e'\Big[-2(1-3\cos^2 I')+6\sin^2 I'\cos 2h+10\sin^2 I'\cos 2g$$

$$+5(1-\cos I')^2\cos(2g-2h)+5(1+\cos I')^2\cos(2g+2h)\Big]\cdot(E'-l)$$

$$+4(1-3\cos^2 I')\Big[3(4+3e'^2)\sin E'-6e'\sin 2E'+e'^2\sin 3E'\Big]$$

$$+2\sin^2 I'\Big[-9(4+3e'^2)\sin(E'-2h)+18e'\sin(2E'-2h)-3e'^2\sin(3E'-2h)$$

$$-9(4+3e'^2)\sin(E'+2h)+18e'\sin(2E'+2h)-3e'^2\sin(3E'+2h)$$

$$-15\{(2+3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(E'-2g)$$

$$+6e'\{1-2(1-e'^2)^{1/2}+(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'-2g)$$

$$-\{(2-3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(3E'-2g)$$

$$-15\{(2+3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(E'+2g)$$

$$+6e'\{1+2(1-e'^2)^{1/2}-(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'+2g)$$

$$-\{(2-3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(3E'+2g)\Big]$$

$$+(1-\cos I')^2\Big[-15\{(2+3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(E'-2g+2h)$$

$$+6e'\{1-2(1-e'^2)^{1/2}+(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'-2g+2h)$$

$$-\{(2-3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(3E'-2g+2h)$$

$$-15\{(2+3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(E'+2g-2h)$$

$$+6e'\{1+2(1-e'^2)^{1/2}-(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'+2g-2h)$$

$$-\{(2-3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(3E'+2g-2h)\Big]$$

$$+(1+\cos I')^2\Big[-15\{(2+3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(E'-2g-2h)$$

$$+6e'\{1-2(1-e'^2)^{1/2}+(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'-2g-2h)$$

$$-\{(2-3e'^2)-2(1-e'^2)^{1/2}+2e'^2(1-e'^2)^{-1/2}\}\sin(3E'-2g-2h)$$

$$-15\{(2+3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(E'+2g+2h)$$

$$+6e'\{1+2(1-e'^2)^{1/2}-(1-e'^2)^{-1/2}(1+e'^2)\}\sin(2E'+2g+2h)$$

$$-\{(2-3e'^2)+2(1-e'^2)^{1/2}-2e'^2(1-e'^2)^{-1/2}\}\sin(3E'+2g+2h)\Big]\Big\}$$

$$+\frac{1}{384}\frac{n'a^2}{\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\Big\{12\Big[-2(1-3\cos^2 I')(2+3e'^2)+6\sin^2 I'(2+3e'^2)\cos 2h$$

$$+30e'^2\sin^2 I'\cos 2g+15e'^2(1-\cos I')^2\cos(2g-2h)+15e'^2(1+\cos I')^2\cos(2g+2h)\Big]$$

$$+12(1-3\cos^2 I')\Big[3e'(4+e'^2)\cos E'-6e'^2\cos 2E'+e'^3\cos 3E'\Big]$$

$$+18\sin^2 I' \left[-3e'(4+e'^2)\cos(E'-2h)+6e'^2\cos(2E'-2h)-e'^3\cos(3E'-2h)\right.$$

$$-3e'(4+e'^2)\cos(E'+2h)+6e'^2\cos(2E'+2h)-e'^3\cos(3E'+2h)$$

$$-5e'\{(2+e'^2)-2(1-e'^2)^{1/2}\}\cos(E'-2g)$$

$$+2\{(2+e'^2)-2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'-2g)$$

$$-e'\{(2-e'^2)-2(1-e'^2)^{1/2}\}\cos(3E'-2g)$$

$$-5e'\{(2+e'^2)+2(1-e'^2)^{1/2}\}\cos(E'+2g)$$

$$+2\{(2+e'^2)+2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'+2g)$$

$$\left.-e'\{(2-e'^2)+2(1-e'^2)^{1/2}\}\cos(3E'+2g)\right]$$

$$+9(1-\cos I')^2\left[-5e'\{(2+e'^2)-2(1-e'^2)^{1/2}\}\cos(E'-2g+2h)\right.$$

$$+2\{(2+e'^2)-2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'-2g+2h)$$

$$-e'\{(2-e'^2)-2(1-e'^2)^{1/2}\}\cos(3E'-2g+2h)$$

$$-5e'\{(2+e'^2)+2(1-e'^2)^{1/2}\}\cos(E'+2g-2h)$$

$$+2\{(2+e'^2)+2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'+2g-2h)$$

$$\left.-e'\{(2-e'^2)+2(1-e'^2)^{1/2}\}\cos(3E'+2g-2h)\right]$$

$$+9(1+\cos I')^2\left[-5e'\{(2+e'^2)-2(1-e'^2)^{1/2}\}\cos(E'-2g-2h)\right.$$

$$+2\{(2+e'^2)-2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'-2g-2h)$$

$$-e'\{(2-e'^2)-2(1-e'^2)^{1/2}\}\cos(3E'-2g-2h)$$

$$- 5\, e' \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \cos (E' + 2g + 2h)$$

$$+ 2\, \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\, (1 + e'^2)\} \cos (2E' + 2g + 2h)$$

$$- e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \cos (3E' + 2g + 2h)\Big]\Big\} \cdot \frac{a'}{r'} \sin E'$$

$$\frac{\partial S_2}{\partial g} = \frac{1}{4}\, \frac{n' R_{\mathbb{C}}^2 J_2}{(1 - e'^2)^{3/2}} \sin^2 I' \, \{3\, e' \cos (f' + 2g) + 3 \cos (2f' + 2g) + e' \cos (3f' + 2g)\}$$

$$+ \frac{1}{4}\, \frac{n' R_{\mathbb{C}}^2 J_{22}}{(1 - e'^2)^{3/2}} \, \{(1 - \cos I')^2 \, [3e' \cos \, (f' + 2g - 2h) + 3 \cos (2f' + 2g - 2h)$$

$$+ e' \cos (3f' + 2g - 2h)]$$

$$+ (1 + \cos I')^2 \, [3e' \cos (f' + 2g + 2h) + 3 \cos (2f' + 2g + 2h) + e' \cos (3f' + 2g + 2h)]\}$$

$$+ \frac{1}{64}\, \frac{n' a^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \Big\{- 60\, e'^2 \, \Big[2 \sin^2 I' \sin 2g + (1 - \cos I')^2 \sin(2g - 2h)$$

$$+ (1 + \cos I')^2 \sin(2g + 2h)\Big] \cdot (E' - l) + 2 \sin^2 I' \, \Big[15 e' \{(2 + e'^2)$$

$$- 2(1 - e'^2)^{1/2}\} \cos (E' - 2g) - 3 \{(2 + e'^2) - 2(1 - e'^2)^{1/2} \, (1 + e'^2)\} \cos (2E' - 2g)$$

$$+ e' \{(2 - e'^2) - 2(1 - e'^2)^{1/2}\} \cos (3E' - 2g)$$

$$- 15\, e' \, \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \cos (E' + 2g)$$

$$+ 3\, \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\, (1 + e'^2)\} \cos (2E' + 2g)$$

$$- e' \, \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \cos (3E' + 2g)\Big]$$

$$+ (1 - \cos I')^2 \, \Big[15\, e' \{(2 + e'^2) - 2(1 - e'^2)^{1/2}\} \cos (E' - 2g + 2h)$$

$$- 3\, \{(2 + e'^2) - 2(1 - e'^2)^{1/2}\, (1 + e'^2)\} \cos (2E' - 2g + 2h)$$

$$+ e' \{(2 - e'^2) - 2(1 - e'^2)^{1/2}\} \cos(3E' - 2g + 2h)$$

$$- 15e' \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \cos(E' + 2g - 2h)$$

$$+ 3 \{(2 + e'^2) + 2(1 - e'^2)^{1/2} (1 + e'^2)\} \cos(2E' + 2g - 2h)$$

$$- e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \cos(3E' + 2g - 2h)\Big]$$

$$+ (1 + \cos I')^2 \Big[15 e' \{(2 + e'^2) - 2(1 - e'^2)^{1/2}\} \cos(E' - 2g - 2h)$$

$$- 3 \{(2 + e'^2) - 2(1 - e'^2)^{1/2} (1 + e'^2)\} \cos(2E' - 2g - 2h)$$

$$+ e' \{(2 - e'^2) - 2(1 - e'^2)^{1/2}\} \cos(3E' - 2g - 2h)$$

$$- 15 e' \{(2 + e'^2) + 2(1 - e'^2)^{1/2}\} \cos(E' + 2g + 2h)$$

$$+ 3 \{(2 + e'^2) + 2(1 - e'^2)^{1/2} (1 + e'^2)\} \cos(2E' + 2g + 2h)$$

$$- e' \{(2 - e'^2) + 2(1 - e'^2)^{1/2}\} \cos(3E' + 2g + 2h)\Big]\Big\}$$

$$\frac{\partial S_2}{\partial h} = \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_{22}}{(1 - e'^2)^{3/2}} \{6 \sin^2 I' [-2(f' - l) \sin 2h - e' \cos(f' - 2h) + e' \cos(f' + 2h)]$$

$$- (1 - \cos I')^2 [3 e' \cos(f' + 2g - 2h) + 3 \cos(2f' + 2g - 2h) + e' \cos(3f' + 2g - 2h)]$$

$$+ (1 + \cos I')^2 [3e' \cos(f' + 2g + 2h) + 3 \cos(2f' + 2g + 2h) + e \ \cos(3f' + 2g + 2h)]\}$$

$$+ \frac{1}{192} \frac{n' a^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \Big\{36 \Big[-2 \sin^2 I' (2 + 3 e'^2) \sin 2h + 5e'^2 (1 - \cos I')^2 \sin(2g - 2h)$$

$$- 5 e'^2 (1 + \cos I')^2 \sin(2g + 2h)\Big] \cdot (E' - l)$$

$$+6\sin^2 I' \Big[9e'(4+e'^2)\cos(E'-2h) - 9e'^2\cos(2E'-2h) + e'^3\cos(3E'-2h)$$

$$-9e'(4+e'^2)\cos(E'+2h) + 9e'^2\cos(2E'+2h) - e'^3\cos(3E'+2h)\Big]$$

$$+3(1-\cos I')^2 \Big[-15e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\cos(E'-2g+2h)$$

$$+3\{(2+e'^2) - 2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'-2g+2h)$$

$$-e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\cos(3E'-2g+2h)$$

$$+15e'\{(2+e'^2) + 2(1-e'^2)^{1/2}\}\cos(E'+2g-2h)$$

$$-3\{(2+e'^2) + 2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'+2g-2h)$$

$$+e'\{(2-e'^2) + 2(1-e'^2)^{1/2}\}\cos(3E'+2g-2h)\Big]$$

$$+3(1+\cos I')^2 \Big[15e'\{(2+e'^2) - 2(1-e'^2)^{1/2}\}\cos(E'-2g-2h)$$

$$-3\{(2+e'^2) - 2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'-2g-2h)$$

$$+e'\{(2-e'^2) - 2(1-e'^2)^{1/2}\}\cos(3E'-2g-2h)$$

$$-15e'\{(2+e'^2) + 2(1-e'^2)^{1/2}\}\cos(E'+2g+2h)$$

$$+3\{(2+e'^2) + 2(1-e'^2)^{1/2}(1+e'^2)\}\cos(2E'+2g+2h)$$

$$-e'\{(2-e'^2) + 2(1-e'^2)^{1/2}\}\cos(3E'+2g+2h)\Big]\Big\}.$$

If

$$\Delta L = L - L'$$

$$\Delta G = G - G'$$

$$\Delta H = H - H',$$

the short periodic perturbations in the Keplerian elements a, e, I are obtained by

$$a = a' + 2a' \frac{\Delta L}{L'}$$

$$e = e' + \frac{1 - e'^2}{e'} \left(\frac{\Delta L}{L'} - \frac{\Delta G}{G'} \right)$$

$$I = I' + \cot I' \left(\frac{\Delta G}{G'} - \frac{\Delta H}{H'} \right)$$

and in the expressions for ΔL, ΔG, ΔH, the variables l, g, h may be replaced by the variables l', g', h' with an error of the 4th order.

It is important to remember the fact that the primed variables are only affected by long periodic perturbations (with an error of the 4th order).

12. The Long Period Terms. Elimination of h'

At this stage the Hamiltonian $F' = F'_0 + F'_1 + F'_2$ depends on the variables g', h', L', G', H', and L' is a constant with respect to time. The next step in von Zeipel's method consists of the elimination of h' and g'. The elimination of h' is performed by means of a generating function

$$S' = L''l' + G''g' + H''h' + S'_1 + S'_2 + \cdots$$

of a canonical transformation from (l', g', h', L', G', H') to (l'', g'', h'', L'', G'', H''). The new Hamiltonian

$$F'' = F''_0 + F''_1 + F''_2 + \cdots$$

should be independent of h''.

In order to simplify the formulas the following symbols are introduced:

$$\mu \equiv \mu_{\mathbb{C}}, b \equiv R_{\mathbb{C}}$$

$$\eta' = \frac{G'}{L'} = \sqrt{1 - e'^2}$$

$$\theta' = \frac{H'}{G'} = \cos I'.$$

Then

$$F' = \frac{\mu^2}{2L'^2} + n_{\mathbb{C}}^* H' + \frac{n_{\mathbb{C}}^2 L'^4}{16\mu^2 \epsilon} \Big\{(5 - 3\eta'^2) \, [(-1 + 3\theta'^2)$$

$$+ 3(1 - \theta'^2) \cos 2h'] + 15(1 - \eta'^2) \left[\frac{1}{2} (1 + \theta')^2 \cos 2(g' + h')\right.$$

$$\left. + (1 - \theta'^2) \cos 2g' + \frac{1}{2} (1 - \theta')^2 \cos 2(g' - h')\right]\Big\}$$

$$+ \frac{1}{4} \frac{\mu^4}{L'^6} \left(\frac{L'}{G'}\right)^3 [-b^2 J_2 (1 - 3\theta'^2) + 6 b^2 J_{22} (1 - \theta'^2) \cos 2h'] .$$

The elimination of h' depends on the solution of the system

$$F_0''(L'') = F_0'(L'') = \frac{\mu^2}{2L''^2}$$

$$F_1''(H'') = F_1'(H'') = n_{\mathbb{C}}^* H''$$

$$F_2'' = F_2' + \frac{\partial S_1'}{\partial h'} \frac{\partial F_1'}{\partial H''} ,$$

- - - - - - - - - - - -

and since S_1' does not depend on l', $L'' = L'$.

As before a particular solution is given by

$$F_2'' = F_{2s}' = \text{part of } F_2' \text{ independent of } h'.$$

$$n_c^* \frac{\partial S_1'}{\partial h'} = -F_{2p}' = -(F_2' - F_{2s}').$$

Then it is found that

$$F_2'' = \frac{n_c^2 L'^4}{16\mu^2 \epsilon} \{(5 - 3\eta''^2)(-1 + 3\theta''^2)$$

$$+ 15(1 - \eta''^2)(1 - \theta''^2) \cos 2g''\} - \frac{1}{4} b^2 J_2 \frac{\mu^4}{L'^6} \frac{1}{\eta''^3} (1 - 3\theta''^2),$$

where

$$\eta'' = \frac{G''}{L'}$$

$$\theta'' = \frac{H''}{G''} .$$

Furthermore,

$$n_c^* S_1' = -\frac{1}{16} \frac{\mu^2}{\epsilon L'^2} \left(\frac{n_c}{n'}\right)^2 \left\{\frac{3}{2} (5 - 3\eta''^2)(1 - \theta''^2) \sin 2h'\right.$$

$$\left.+ \frac{15}{4} (1 - \eta''^2) \left[(1 + \theta'')^2 \sin 2(g' + h') - (1 - \theta'')^2 \sin 2(g' - h')\right]\right\}$$

$$- \frac{3}{4} n'^2 b^2 J_{22} \frac{1}{\eta''^3} (1 - \theta''^2) \sin 2h'$$

The long periodic perturbations depending on the motion of the node (and eventually on that of the pericenter) are given by

$$L' = L'' + \frac{\partial S_1'}{\partial l'} = L''$$

$$G' = G'' + \frac{\partial S_1'}{\partial g'} =$$

$$= G'' \left\{1 - \frac{15}{32\epsilon}\left(\frac{n_{☾}^2}{n_{☾}^* n'}\right) \frac{1-\eta''^2}{\eta''} \left[(1+\theta'')^2 \cos 2(g''+h'') - (1-\theta'')^2 \cos 2(g''-h'')\right]\right\}$$

$$H' = H'' + \frac{\partial S_1'}{\partial h'} =$$

$$= H'' \left\{1 - \frac{3}{16\epsilon}\left(\frac{n_{☾}^2}{n_{☾}^* n'}\right) \frac{1}{\eta''\theta''} \left[(5 - 3\eta''^2)(1-\theta''^2)\cos 2h''\right.\right.$$

$$+ \frac{5}{2}(1-\eta''^2)\left\{(1+\theta'')^2 \cos 2(g''+h'')\right.$$

$$\left.\left.\left. + (1-\theta'')^2 \cos 2(g''-h'')\right\}\right] - \frac{3}{2}\frac{n'}{H''}\left(\frac{n'}{n_{☾}^*}\right) b^2 J_{22} \frac{1-\theta''^2}{\eta''^3} \cos 2h''\right\}$$

$$l' = l'' - \frac{\partial S_1'}{\partial L''} =$$

$$= l'' + \frac{1}{16\epsilon}\left(\frac{n_{☾}^2}{n_{☾}^* n'}\right)\left\{3(10 - 3\eta''^2)(1-\theta''^2)\sin 2h''\right.$$

$$\left. + 15\left(1 - \frac{1}{2}\eta''^2\right)\left[(1+\theta'')^2 \sin 2(g''+h'') - (1-\theta'')^2 \sin 2(g''-h'')\right]\right\}$$

$$- \frac{9}{4}\frac{n'^2}{n_{☾}^*} b^2 J_{22} \frac{1-\theta''^2}{\eta''^3 L'} \sin 2h''$$

$$g' = g'' - \frac{\partial S_1'}{\partial G''} =$$

$$= g'' + \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\frac{1}{\eta''}\Bigg\{3(5\theta''^2 - 3\eta''^2)\ \sin 2h''$$

$$- \frac{15}{2}(1 + \theta'')(\eta''^2 + \theta'')\ \sin 2(g'' + h'')$$

$$+ \frac{15}{2}(1 - \theta'')(\eta''^2 - \theta'')\ \sin 2(g'' - h'')\Bigg\}$$

$$- \frac{3\,n'^2}{4\,n_{\mathbb{C}}^*}\, b^2 J_{22}\, \frac{3 - 5\theta''^2}{G''\,\eta''^3}\ \sin 2h''$$

$$h' = h'' - \frac{\partial S_1'}{\partial H''} =$$

$$= h'' + \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\frac{1}{\eta''}\Bigg\{3\theta''(3\eta''^2 - 5)\ \sin 2h''$$

$$+ \frac{15}{2}(1 - \eta''^2)\Big[(1 + \theta'')\ \sin 2(g'' + h'') + (1 - \theta'')\ \sin 2(g'' - h'')\Big]\Bigg\}$$

$$- \frac{3}{2}\frac{n'^2}{n_{\mathbb{C}}^*}\, b^2 J_{22}\, \frac{\theta''^2}{H''\,\eta''^3}\ \sin 2h''.$$

It is important to note that these terms are factored by 1st order factors. Then, a second order theory produces first order long period terms. It is necessary to go to 3rd order to obtain second order long period terms. This is done next.

13. The Second Order Long Period Terms and Elimination of h' and the Time

Care must be exercised in the evaluation of the third order part of the Hamiltonian. One of the dangers involved is including in the third order part of the Hamiltonian terms which are really second order. This difficulty could arise, for example, in the evaluation of the solar perturbations. This would result in the introduction of more terms into the second order Hamiltonian, which is permissible.

An additional complication is the uncertainty of the values of the zonal harmonics for the potential field of the moon. There is no harm done in the theory if J_2 and J_{22} are really smaller than the values assigned to them. However, if J_4 is introduced and treated as though it were of order J_2^2 when it is really greater than J_2^2, then factors of the form J_4/J_2 might invalidate the theory.

Keeping this in mind, one can list the small parameters of 3rd order:

J_3, J_4, J_5	(moon's potential field)
$j_2\left(\frac{n_{☾}}{n}\right)^2$	(earth's oblateness)
$\left(\frac{n_{\oplus}}{n}\right)^2$	(sun's perturbation)
$\left(\frac{n_{☾}}{n}\right)^2 \sin\frac{i_{☾}}{2}$	(correction due to the inclination of the moon's orbit to its equator)
σ	(solar radiation)

$$\left(\frac{n_{☾}}{n}\right)^2 e_{☾} \qquad \text{(eccentricity of the moon's orbit)}$$

$$\left(\frac{n_{☾}}{n}\right)^3 \qquad \text{(earth's perturbations - 3rd Legendre polynomial)}$$

$$\frac{\alpha n_{\odot}}{n} \qquad \text{(correction due to physical libration)}$$

Since short periodic perturbations of the third order will be neglected, it is understood that all terms of third order which depend on l will be ignored. Suppose that F_3 is the third order part of the Hamiltonian and F_3' the part free of short period terms. The Hamiltonian will be time dependent through the longitudes of the earth and of the sun. This fact introduces one more degree of freedom in the problem and it is necessary to introduce the time as a canonical variable. This is done through use of the following equations (τ = t + const.):

$$\frac{d\tau}{dt} = -\frac{\partial \mathcal{F}}{\partial T} = 1$$

$$\frac{dT}{dt} = +\frac{\partial \mathcal{F}}{\partial \tau} = \frac{\partial \mathcal{F}}{\partial t} = \frac{\partial F_3}{\partial t} ,$$

so that

$$\mathcal{F} = F - T$$

and

$$T = -F_3 + \text{const}.$$

This additional complication does not affect the development of the theory up to this point, if one sets

$$\frac{\partial S_2'}{\partial \tau} = 0 .$$

The next step is then to eliminate h' to 3rd order to obtain 2nd order long period perturbations depending on the argument h'.

The orders of magnitude are as follows:

$$\mathcal{F}_0' = F_0' - T = \frac{\mu^2}{2L'} - T$$

$$\mathcal{F}_1' = F_1' = n_{\mathbb{C}}^* H'$$

$$\mathcal{F}_2' = F_2'$$

$$\mathcal{F}_3' = F_3' ,$$

where T has been incorporated into $\mathcal{F}_0'$ for reasons that will become obvious.

The von Zeipel differential equation of the 3rd order is then

$$\mathcal{F}_3' + \frac{\partial S_1'}{\partial h'} \frac{\partial \mathcal{F}_2'}{\partial H''} + \frac{\partial S_1'}{\partial g'} \frac{\partial \mathcal{F}_2'}{\partial G''} + \frac{\partial S_2'}{\partial h'} \frac{\partial \mathcal{F}_1'}{\partial H''} + \frac{\partial S_3'}{\partial \tau} \frac{\partial \mathcal{F}_0'}{\partial T}$$

$$= \mathcal{F}_3'' + \frac{\partial S_1'}{\partial G''} \frac{\partial \mathcal{F}_2''}{\partial g'} ,$$

or

$$F_3' + \frac{\partial S_1'}{\partial h'} \frac{\partial F_2'}{\partial H''} + \frac{\partial F_2'}{\partial G''} \frac{\partial S_1'}{\partial g'} + n_{\mathbb{C}}^* \frac{\partial S_2'}{\partial h'} - \frac{\partial S_3'}{\partial \tau}$$

$$= F_3'' + \frac{\partial F_2''}{\partial g'} \frac{\partial S_1'}{\partial G''} .$$

Since we had supposed that $\partial S_2'/\partial\tau = 0$ in the previous order evaluation, a particular solution of the 3rd order equation is found by taking

$$F_3'' = F_{3s}' + \left(\frac{\partial S_1'}{\partial h'}\frac{\partial F_2'}{\partial H''}\right)_s + \left(\frac{\partial F_2'}{\partial G''}\frac{\partial S_1'}{\partial g'}\right)_s - \left(\frac{\partial F_2''}{\partial g'}\frac{\partial S_1'}{\partial G''}\right)_s ,$$

where the subscript "s" designates non-dependence upon h'' and τ. Further,

$$n_{\mathfrak{C}}^* \frac{\partial S_2'}{\partial h'} - F_{3ph'}' - \left(\frac{\partial S_1'}{\partial h'}\quad\frac{\partial F_2'}{\partial H''}\right)_{ph'}$$

$$- \left(\frac{\partial F_2'}{\partial G''}\quad\frac{\partial S_1'}{\partial g'}\right)_{ph'} + \left(\frac{\partial F_2''}{\partial g'}\quad\frac{\partial S_1'}{\partial G''}\right)_{ph'}$$

and

$$\frac{\partial S_3'}{\partial\tau} = F_{3p\tau}' + \left(\frac{\partial S_1'}{\partial h'}\quad\frac{\partial F_2'}{\partial H''}\right)_{p\tau} + \left(\frac{\partial F_2'}{\partial G''}\quad\frac{\partial S_1'}{\partial g'}\right)_{p\tau}$$

$$- \left(\frac{\partial F_2''}{\partial g'}\quad\frac{\partial S_1'}{\partial G''}\right)_{p\tau} .$$

where the subscript ph' indicates the inclusion of all trigonometric terms with arguments of the form $jh' + kg'$, j and k integers $(j \neq 0)$, and the subscript $p\tau$ indicates the inclusion of all trigonometric terms with arguments of the form $j\tau + kh' + mg'$, where j, k, and m are integers $(j \neq 0)$.

Then S_2' will be obtained as a periodic function independent of the time, and S_3' as a periodic function dependent on time.

Since the functions F'_{2p}, F'_2, S'_1, and F''_2 are known, the only parts that have to be computed are those corresponding to the 3rd order terms of the Hamiltonian (F'_3).

14. Third Order Terms Generated by Coupling of 2nd Order Terms

It was found that

$$F'_2 = \frac{1}{16\epsilon}\left(\frac{n_{\text{¢}}}{n'}\right)^2 n'L' \left\{(5 - 3\eta'^2)\,[(-1 + 3\theta'^2) + 3(1 - \theta'^2)\cos 2h']\right.$$

$$+ 15(1 - \eta'^2)\left[\frac{1}{2}(1 + \theta')^2 \cos 2(g' + h') + (1 - \theta'^2)\cos 2g'\right.$$

$$\left.\left. + \frac{1}{2}(1 - \theta')^2 \cos 2(g' - h')\right]\right\} + \frac{1}{4}\,\frac{\mu^4}{L'^6}\left(\frac{L'}{G'}\right)^3 \,[-b^2 J_2 (1 - 3\theta'^2)$$

$$+ 6b^2 J_{22}\,(1 - \theta'^2)\cos 2h']$$

$$F''_2 = \frac{1}{16\epsilon}\left(\frac{n_{\text{¢}}}{n'}\right)^2 n'L' \;\{(5 - 3\eta''^2)\,(-1 + 3\theta''^2)$$

$$+ 15(1 - \eta''^2)\,(1 - \theta''^2)\cos 2g''\} - \frac{1}{4}\,b^2 J_2\,\frac{\mu^4}{L'^6}\,\frac{1}{\eta''^3}\,(1 - 3\theta''^2)$$

and

$$-n_{\mathbb{C}}^{*}S_1' = -\frac{1}{16}\frac{\mu^2}{\epsilon L'^2}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\left\{\frac{3}{2}(5-3\eta''^2)(1-\theta''^2)\sin 2h'\right.$$

$$\left.+\frac{15}{4}(1-\eta''^2)\left[(1+\theta'')^2\sin 2(g'+h')-(1-\theta'')^2\sin 2(g'-h')\right]\right\}$$

$$-\frac{3}{4}n'^2 b^2 J_{22}\frac{1}{\eta''^3}(1-\theta''^2)\sin 2h',$$

and in the coupling terms the variables in F_2' can be double-primed with an error that is of the 4th order at least.

The partials needed are given below.

$$\frac{\partial F_2'}{\partial H''} = \frac{3}{16\epsilon}n'\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\frac{1}{\eta''\theta''}\{2(5-3\eta''^2)\theta''^2(1-\cos 2h'')$$

$$+5(1-\eta''^2)\theta''[(1+\theta'')\cos 2(g''+h'')-2\theta''\cos 2g''$$

$$-(1-\theta'')\cos 2(g''-h'')]\}+\frac{3}{2}b^2 n'^2\frac{\theta''^2}{H''\eta''^3}[J_2-2J_{22}\cos 2h'']$$

$$\frac{\partial F_2'}{\partial G''} = \frac{3n'}{16\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\frac{1}{\eta''}\{2(\eta''^2-5\theta''^2)+2(5\theta''^2-3\eta''^2)\cos 2h''$$

$$-5(1+\theta'')(\eta''^2+\theta)\cos 2(g''+h'')+10(\theta''^2-\eta''^2)\cos 2g''$$

$$+5(1-\theta'')(\theta''-\eta''^2)\cos 2(g''-h'')\}+\frac{3}{4}\frac{n'^2 b^2}{G''\eta''^3}\{J_2(1-5\theta''^2)$$

$$-2J_{22}(3-5\theta''^2)\cos 2h''\}$$

$$\frac{\partial S_1'}{\partial h'} = \frac{1}{n_{\mathbb{C}}^*} F_{2p}' = -\frac{3L'}{32\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \{2(5 - 3\eta''^2)(1 - \theta''^2)\cos 2h''$$

$$+ 5(1 - \eta''^2)[(1 + \theta'')^2 \cos 2(g'' + h'') + (1 - \theta'')^2 \cos 2(g'' - h'')]\}$$

$$- \frac{3}{2}\frac{n'^2}{n_{\mathbb{C}}^*} b^2 J_{22} \frac{1 - \theta''^2}{\eta''^3} \cos 2h''$$

$$\frac{\partial S_1'}{\partial g'} = -\frac{15}{32\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \frac{1 - \eta''^2}{\eta''} G'' [(1 + \theta'')^2 \cos 2(g'' + h'')$$

$$- (1 - \theta'')^2 \cos 2(g'' - h'')]$$

$$\frac{\partial F_2''}{\partial g'} = -\frac{15}{8\epsilon} n'L'\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 (1 - \eta''^2)(1 - \theta''^2)\sin 2g''$$

$$\frac{\partial S_1'}{\partial G''} = -\frac{3}{32\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\frac{1}{\eta''} \{2(5\theta''^2 - 3\eta''^2)\sin 2h''$$

$$- 5(1 + \theta'')(\eta''^2 + \theta'')\sin 2(g'' + h'') + 5(1 - \theta'')(\eta''^2 - \theta'') \sin 2(g'' - h'')\}$$

$$- \frac{3}{4}\frac{n'^2 b^2 J_{22}}{n_{\mathbb{C}}^* \eta''^3 G''}(-3 + 5\theta''^2)\sin 2h''.$$

From these equations it can be shown that

$$S_2'(\text{coupling}) = +\frac{9}{4096\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 \frac{L'}{\eta''} \{-32\theta''(1 - \theta''^2)\eta''^2(15 - 17\eta''^2)\sin 2h''$$

$$- 4\theta''(1 - \theta''^2)(5 - 3\eta''^2)^2 \sin 4h''$$

$$+ 80(1 + \theta'')^2(2 - 3\theta'')\eta''^2(1 - \eta''^2)\sin(2g'' + 2h'')$$

$$+80(1-\theta'')^2(2+3\theta'')\eta''^2(1-\eta''^2)\sin(2g''-2h'')$$

$$+10(1+\theta'')^2(1-\theta'')(1-\eta''^2)[5(1-\theta'')-6\eta''^2]\sin(2g''+4h'')$$

$$+10(1-\theta'')^2(1+\theta'')(1-\eta''^2)[5(1+\theta'')-6\eta''^2]\sin(2g''-4h'')$$

$$+25(1+\theta'')^3(1-\eta''^2)[(1-\theta'')-2\eta''^2]\sin(4g''+4h'')$$

$$+25(1-\theta'')^3(1-\eta''^2)[(1+\theta'')-2\eta''^2]\sin(4g''-4h'')\}$$

$$+\frac{9}{256\epsilon}n'\eta''^{-4}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_2 b^2\ \{4\theta''(1-\theta''^2)(5-3\eta''^2)\sin 2h''$$

$$+5(1+\theta'')^2(1+2\theta''-5\theta''^2)(1-\eta''^2)\sin(2g''+2h'')$$

$$+5(1-\theta'')^2(1-2\theta''-5\theta''^2)(1-\eta''^2)\sin(2g''-2h'')\}$$

$$-\frac{9}{512\epsilon}n'\eta''^{-4}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_{22}b^2\{-8\theta''(1-\theta''^2)(5-3\eta''^2)[2\sin 2h''-\sin 4h'']-$$

$$-20(1+\theta'')^2(1-\theta'')(3-5\theta'')(1-\eta''^2)\sin(2g''+2h'')$$

$$-20(1-\theta'')^2(1+\theta'')(3+5\theta'')(1-\eta''^2)\sin(2g''-2h'')$$

$$+5(1+\theta'')^2(1-\theta'')(1+5\theta'')(1-\eta''^2)\sin(2g''+4h'')$$

$$+5(1-\theta'')^2(1+\theta'')(1-5\theta'')(1-\eta''^2)\sin(2g''-4h'')\}$$

$$+\frac{9}{8}\left(\frac{n'}{n^*_{\mathbb{C}}}\right)^2 n'^2\ \frac{\theta''^2(1-\theta''^2)}{\eta''^6 H''}J_2J_{22}b^4\sin 2h''-\frac{9}{16}\left(\frac{n'}{n^*_{\mathbb{C}}}\right)^2 n'^2\ \frac{\theta''^2(1-\theta''^2)}{\eta''^6 H''}\ J^2_{22}\ b^4\ \sin 4h'',$$

$$F_3''(\text{coupling}) = +\frac{9}{128}\left(\frac{n_{\mathbb{C}}}{n'}\right)^3\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\frac{n^2}{\epsilon^2}\, a'^2\sqrt{1-e''^2}\cos I''\{[(2+33e''^2)-(2-17e''^2)\cos^2 I'']$$

$$+15e''^2\sin^2 I''\cos 2g''\}$$

$$+\frac{9}{32}\left(\frac{n_{\mathbb{C}}}{n'}\right)\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\frac{n'^2\sin^2 I''\cos I''}{\epsilon(1-e''^2)^2}\, J_{22}\, b^2\,\{2(2+3e''^2)+15e''^2\cos 2g''\}$$

$$+\frac{9}{4}\left(\frac{n'}{n_{\mathbb{C}}^*}\right)\frac{n'^2\sin^2 I''\cos I''}{a'^2(1-e''^2)^{7/2}}\, J_{22}^2\, b^4,$$

and, of course,

$$S_3'\ (\text{coupling}) = 0.$$

It is important to note that S_2' (coupling) is made up of second order terms though it is obtained by means of multiplying a second order quantity by a first order quantity.

The next few sections are devoted to the computation of F_3'. From that computation various additional parts of F_3'', S_3', and S_2' are then derived.

15. The Radiation Pressure

For this computation the shadow effect will be neglected. Its inclusion would require, of course, the introduction of a shadow function due both to the moon and the earth.

Consider σ to be the absolute value of the acceleration of the orbiter arising from the solar radiation. Then, the disturbing function for the radiation pressure will be given by

$$R_\sigma = -\sigma r \cos S_{13}'',$$

or, neglecting the inclination of the lunar equator to the sun's orbit,

$$R_\sigma = -\sigma r \, [\cos(f+g)\cos(\Omega - \lambda_\odot)$$

$$-\sin(f+g)\sin(\Omega-\lambda_\odot)\cos I] \, .$$

The elimination of short periodic terms gives

$$R'_\sigma = \frac{3}{2}\,\sigma a' e' \left\{\cos^2 \frac{I'}{2}\cos(g'+h'+\lambda_\oplus-\lambda_\odot) + \sin^2\frac{I'}{2}\cos(g'-h'+\lambda_\odot-\lambda_\oplus)\right\},$$

where use was made of the formulas

$$\frac{1}{2\pi}\int_0^{2\pi} r\cos f\, dl = -\frac{3}{2}\, a e$$

$$\frac{1}{2\pi}\int_0^{2\pi} r\sin f\, dl = 0,$$

and where the node Ω was written in terms of the canonical variable h. There is no secular contribution from this effect, that is, F'_{3s} (radiation) $= R'_{\sigma s} = 0$. On the other hand, since there are no terms strictly independent of time, the contribution to S'_2 is zero, or

$$S'_2 \,(\text{radiation}) = 0.$$

Therefore,

$$\frac{\partial S'_3\,(\text{radiation})}{\partial\tau} = F'_{3p\tau}\;(\text{radiation}) = R'_\sigma\,,$$

or

$$S'_3\,(\text{radiation}) = +\frac{3}{2}\,\frac{\sigma a'' e''}{n^*_☾ - n^*_\oplus}\left\{\cos^2\frac{I''}{2}\sin(g''+h''+\lambda_\oplus-\lambda_\odot)\right.$$

$$\left. -\sin^2\frac{I''}{2}\sin(g''-h''+\lambda_\odot-\lambda_\oplus)\right\},$$

where

$$\lambda_{\odot} = n_{\oplus}^{*} \tau + \text{const.}$$

$$\lambda_{\oplus} = n_{\mathbb{C}}^{*} \tau + \text{const.},$$

and $n_{\oplus}^{*}$ and $n_{\mathbb{C}}^{*}$ are, respectively, the mean motion in longitude of the sun around the moon and of the moon around the earth. Of course, $n_{\mathbb{C}}^{*} - n_{\odot}^{*} \simeq n_{\mathbb{C}}^{*}$ is a good approximation.

16. The Second Legendre Polynomial for the Sun's Gravitational Perturbations

To restate what has been said previously, the most important term in the disturbing function due to the sun's gravity is

$$\frac{\mu_3}{r_3} \left(\frac{r}{r_3}\right)^2 P_2 (\cos S_{13}''),$$

where, for the computation of S_{13}'', the sun can be considered to move along the moon's equator.

Then

$$\cos S_{13}'' = \cos (f + g) \cos (\Omega - \lambda_{\odot}) - \sin (f + g) \sin (\Omega - \lambda_{\odot}) \cos I.$$

On the other hand, neglecting the mass of the moon,

$$n_{\oplus}^2 a_3^3 = k^2 (m_{\odot} + m_{\oplus}) = k^2 m_{\odot} \left(1 + \frac{m_{\oplus}}{m_{\odot}}\right),$$

and since the ratio $m_{\oplus}/m_{\odot}$ can be neglected,

$$n_{\oplus}^2 a_3^3 = \mu_3 .$$

Furthermore, if the eccentricity of the earth orbit is neglected,

$$F_3\ (\text{sun}) = n_\oplus^2\, r^2\, P_2\, (\cos S''_{13}) =$$

$$= \frac{1}{2}\, n_\oplus^2\, r^2\, (3 \cos^2 S''_{13} - 1) =$$

$$= \frac{3}{2}\, n_\oplus^2\, r^2\, \{\cos^2 (f+g) \cos^2 (\Omega - \lambda_\odot) - 2 \sin (f+g) \cos (f+g) .$$

$$\cdot \sin (\Omega - \lambda_\odot) \cos (\Omega - \lambda_\odot) \cos I + \sin^2 (f+g) \sin^2 (\Omega - \lambda_\odot) \cos^2 I\}$$

$$- \frac{1}{2}\, n_\oplus^2\, r^2 .$$

That this contribution is of the third order is visible from the fact that

$$n_\oplus^2 = \left(\frac{n_\oplus}{n}\right)^2 n^2 \simeq \left(\frac{1}{10}\, \frac{n_☾}{n}\right)^2\ n^2,$$

where

$$\frac{n_\oplus}{n_☾} \approx \frac{28}{365} .$$

Now

$$F'_3\ (\text{sun}) = \frac{3}{2}\, n_\oplus^2\, r^2\, \{(\cos^2 f \cos^2 g + \sin^2 f \sin^2 g) \cos^2 (\Omega - \lambda_\odot)$$

$$- 2 (- \sin^2 f \sin g \cos g + \cos^2 f \sin g \cos g) \sin (\Omega - \lambda_\odot) \cos (\Omega - \lambda_\odot) \cos I$$

$$+ (\sin^2 f \cos^2 g + \cos^2 f \sin^2 g) \sin^2 (\Omega - \lambda_\odot) \cos^2 I\}$$

$$- \frac{1}{2}\, n_\oplus^2\, r^2 + \frac{3}{2}\, n_\oplus^2\, r^2 \sin f \cos f \cdot Q ,$$

where Q is independent of f.

In order to obtain the part of F_3 (sun) independent of l, F_3 (sun) is averaged over l. The following well known integrals are used:

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \cos^2 f \, dl = a^2 \left(\frac{1}{2} + 2e^2\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \sin f \cos^n f \, dl = 0 \qquad (n = 0, 1, 2, \ldots)$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \sin^2 f \, dl = \frac{1}{2} a^2 (1 - e^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \, dl = a^2 \left(1 + \frac{3}{2} e^2\right) .$$

They are obtained using the eccentric anomaly as the variable of integration. It follows that

$$\begin{aligned} F_3' (\text{sun}) = \frac{1}{32} n_\oplus^2 a''^2 \Big\{ & -2 (2 + 3e''^2)(1 - 3\cos^2 I'') + 30 e''^2 \sin^2 I'' \cos 2g'' \\ & + 6 (2 + 3e''^2) \sin^2 I'' \cos 2 (h'' + \lambda_\oplus - \lambda_\odot) \\ & + 15 e''^2 (1 + \cos I'')^2 \cos 2 (g'' + h'' + \lambda_\oplus - \lambda_\odot) \\ & + 15 e''^2 (1 - \cos I'')^2 \cos 2 (g'' - h'' - \lambda_\oplus + \lambda_\odot) \Big\} . \end{aligned}$$

The secular contribution is

$$F_3'' (\text{sun}) = \frac{1}{16} n_\oplus^2 a''^2 \left\{ -(2 + 3e''^2)(1 - 3\cos^2 I'') + 15 e''^2 (1 - \cos^2 I'') \cos 2g'' \right\} .$$

There is no contribution to S_2', and the contribution to S_3' is

$$S_3'\ (\text{sun}) = \frac{3}{64} \frac{n_\oplus^2 a''^2}{n_☾^* - n_\oplus^*} \{2\,(2+3e^2) \sin^2 I'' \sin 2\,(h'' + \lambda_\oplus - \lambda_\odot)$$

$$+ 5e''^2 (1 + \cos I'')^2 \sin 2\,(h'' + g'' + \lambda_\oplus - \lambda_\odot)$$

$$+ 5e''^2 (1 - \cos I'')^2 \sin 2\,(h'' - g'' + \lambda_\oplus - \lambda_\odot)\} .$$

17. The Third Legendre Polynomial for the Earth's Gravitational Perturbations

Referring to page (19)

$$F_3\ (\text{earth}) = \frac{\mu_2}{r_0} \left(\frac{r}{r_0}\right)^3 P_3\ (\cos S_{10}') .$$

Now,

$$n_☾^2\, a_☾^3 = k^2\, (m_\oplus + m_☾) = k^2\, m_\oplus \left(1 + \frac{m_☾}{m_\oplus}\right) = \mu_2\, \epsilon,$$

so that considering the moon's orbit to be circular,

$$F_3\ (\text{earth}) = \frac{n_☾^2\, a_☾^3}{\epsilon} \frac{r^3}{a_☾^4} P_3\ (\cos S_{10}') =$$

$$= \frac{n_☾^2}{\epsilon\, a_☾} r^3\, P_3\ (\cos S_{10}') =$$

$$= \frac{1}{2} \frac{n_☾^2}{\epsilon\, a_☾} r^3\, (5 \cos^3 S_{10}' - 3 \cos S_{10}'),$$

where it is permissible to write

$$\cos \tilde{S}'_{10} = \cos (f+g) \cos (\Omega - \lambda_{\oplus}) - \sin (f+g) \sin (\Omega - \lambda_{\oplus}) \cos I =$$

$$= \frac{1}{2} \cos f \left[(1+\cos I) \cos (g+h) + (1-\cos I) \cos (g-h)\right]$$

$$- \frac{1}{2} \sin f \left[(1+\cos I) \sin (g+h) + (1-\cos I) \sin (g-h)\right].$$

If

$$A_{\oplus} = \frac{1}{2} \left[(1+\cos I) \cos (g+h) + (1-\cos I) \cos (g-h)\right]$$

$$B_{\oplus} = \frac{1}{2} \left[(1+\cos I) \sin (g+h) + (1-\cos I) \sin (g-h)\right],$$

then

$$\cos S'_{10} = A_{\oplus} \cos f - B_{\oplus} \sin f,$$

so that

$$F_3 \text{ (earth)} = \frac{1}{2} \frac{n_{☾}^2}{\epsilon\, a_{☾}} r^3 \left\{5 \left[A_{\oplus}^3 \cos^3 f - 3A_{\oplus}^2 B_{\oplus} \cos^2 f \sin f\right.\right.$$

$$\left.\left. + 3A_{\oplus} B_{\oplus}^2 \cos f \sin^2 f - B_{\oplus}^3 \sin^3 f\right] - 3 \left[A_{\oplus} \cos f - B_{\oplus} \sin f\right]\right\}.$$

The short period terms are eliminated with the aid of the integrals

$$\frac{1}{2\pi} \int_0^{2\pi} r^3 \cos^3 f \, dl = - a^3 \left(\frac{15}{8} e + \frac{5}{2} e^3\right) = - \frac{5}{8} a^3 e (3 + 4 e^2)$$

$$\frac{1}{2\pi} \int_0^{2\pi} r^3 \sin f \cos^n f \, dl = 0 \qquad (n = 0, 1, \ldots)$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^3 \cos f \sin^2 f \, dl = -\frac{5}{8} a^3 e (1-e^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^3 \cos f \, dl = -\frac{5}{8} a^3 e (4+3e^2) \, .$$

It follows that

$$F_3' (\text{earth}) = -\frac{5}{16} \frac{n_{\mathbb{C}}^2 a'^3 e'}{\epsilon a_{\mathbb{C}}} \left\{ 5 \left[(3+4e'^2) A_{\oplus}'^3 + 3 A_{\oplus}' B_{\oplus}'^2 \quad (1-e'^2) \right] \right.$$

$$\left. - 3 A_{\oplus}' (4+3e'^2) \right\} ,$$

where

$$A_{\oplus}'^3 = \frac{1}{32} \left\{ 3 (1-\theta'^2)(1-\theta') \cos(g'-3h') + (1-\theta')^3 \cos(3g'-3h') \right.$$

$$+ 3 (1-\theta'^2)(1-\theta') \cos(3g'-h') + 3 (1-\theta')(3+2\theta'+3\theta'^2) \cos(g'-h')$$

$$+ 3 (1+\theta')(3-2\theta'+3\theta'^2) \cos(g'+h') + 3 (1-\theta'^2)(1+\theta') \cos(3g'+h')$$

$$\left. + 3 (1-\theta'^2)(1+\theta') \cos(g'+3h') + (1+\theta')^3 \cos(3g'+3h') \right\}$$

$$A_{\oplus}' = \frac{1}{2}(1+\theta') \cos(g'+h') + \frac{1}{2}(1-\theta') \cos(g'-h')$$

$$A_{\oplus}' B_{\oplus}'^2 = \frac{1}{32} \left\{ (1+\theta')(3-2\theta'+3\theta'^2) \cos(g'+h') \right.$$

$$+ (1-\theta')(3+2\theta'+3\theta'^2) \cos(g'-h') - 3 (1-\theta')(1-\theta'^2) \cos(3g'-h')$$

$$+ (1-\theta')(1-\theta'^2) \cos(g'-3h') - (1-\theta')^3 \cos(3g'-3h')$$

$$- (1+\theta')^3 \cos(3g'+3h') - 3(1+\theta')(1-\theta'^2)\cos(3g'+h')$$

$$+ (1+\theta')(1-\theta'^2)\cos(g'+3h')\Big\} .$$

The contributions to F_3'' and to S_3' are both zero. The contribution to S_2' is then given by

$$S_2' \text{ (earth)} = -\frac{1}{n_{☾}^*}\int F_3' \text{ (earth) } dh',$$

or

$$S_2' \text{ (earth)} = \frac{5}{1536}\left(\frac{n_{☾}}{n_{☾}^*}\right)\frac{n_{☾} a''^3 e''}{a_{☾}\,\epsilon}\Big\{-9(1+\cos I'')(1+10\cos I''-15\cos^2 I'')(4+3e''^2)\sin(g''+h'')$$

$$+9(1-\cos I'')(1-10\cos I''-15\cos^2 I'')(4+3e''^2)\sin(g''-h'')$$

$$+15\sin^2 I''(1+\cos I'')(4+3e''^2)\sin(g''+3h'')$$

$$-15\sin^2 I''(1-\cos I'')(4+3e''^2)\sin(g''-3h'')$$

$$+315\sin^2 I''(1+\cos I'')\,e''^2\sin(3g''+h'')$$

$$-315\sin^2 I''(1-\cos I'')\,e''^2\sin(3g''-h'')$$

$$+35(1+\cos I'')^3 e''^2\sin(3g''+3h'')$$

$$-35(1-\cos I'')^3 e''^2\sin(3g''-3h'')\Big\} .$$

18. The Eccentricity of the Moon's Orbit

The correction for the eccentricity of the moon's orbit is given by

$$F_3(e_{☾}) = \frac{n_{☾}^2}{\epsilon}\left[\left(\frac{a_{☾}}{r_{☾}}\right)^3 - 1\right] r^2 P_2(\cos\tilde{S}_{10}),$$

or, keeping only the first power of $e_{☾}$,

$$F_3 (e_{☾}) = \frac{3}{2} e_{☾} \frac{n_{☾}^2}{\epsilon} r^2 \cos l_{\oplus} P_2 (\cos \tilde{S}_{10})$$

$$= \frac{3}{4} \frac{e_{☾} n_{☾}^2}{\epsilon} r^2 (3 \cos^2 \tilde{S}_{10} - 1) \cos l_{\oplus} ,$$

where the mean anomaly $l_{\oplus}$ is $l_{\oplus} = n_{☾} t + \text{const}$. Then,

$$F_3' (e_{☾}) = \frac{3}{64} \frac{n_{☾}^2 a'^2}{\epsilon} e_{☾} \{- 2 (2+ 3e'^2)(1- 3 \cos^2 I') + 30 e'^2 \sin^2 I' \cos 2g'$$

$$+ 6 (2+ 3e'^2) \sin^2 I' \cos 2h' + 15e'^2 (1+\cos I')^2 \cos (2g' + 2h')$$

$$+ 15 e'^2 (1- \cos I')^2 \cos (2g' - 2h')\} \cos l_{\oplus} .$$

There are no contributions to S_2' or F_3''. The contribution to S_3' is

$$S_3' (e_{☾}) = \frac{3}{64} \frac{n_{☾} a''^2 e_{☾}}{\epsilon} \{- 2 (2+3 e''^2) (1- 3 \cos^2 I'')$$

$$+ 30 e''^2 \sin^2 I'' \cos 2g'' + 6 (2+ 3 e''^2) \sin^2 I'' \cos 2h''$$

$$+ 15 e''^2 (1+ \cos I'')^2 \cos (2g'' + 2h'')$$

$$+ 15 e''^2 (1- \cos I'')^2 \cos (2g'' - 2h'')\} \sin l_{\oplus} .$$

It is worthwhile to note that although this result is a part of a third order generating function its actual order is $(n_{☾}/n) e_{☾}$ which is considered to be second order. A small divisor $n_{☾}$ is introduced through the integration.

19. The Inclination of the Moon's Orbit to its Equator

The expression for the earth's perturbation is

$$\frac{n_{☾}^2 r^2}{\epsilon} P_2 (\cos S'_{10}),$$

where $\cos S'_{10} = K(i_{☾})$. By a Taylor expansion, if $\sin i_{☾}$ is small,

$$\cos S'_{10} \simeq K(0) + \left\{\frac{\partial K(i_{☾})}{\partial (\sin i_{☾})}\right\}_{i_{☾}=0} \sin i_{☾} =$$

$$= \cos \tilde{S}'_{10} + \left\{\frac{\partial \cos S'_{10}}{\cos i_{☾} \, \partial i_{☾}}\right\}_{i_{☾}=0} \sin i_{☾} .$$

Thus the correction to be introduced is

$$F_3(i_{☾}) = \frac{3 n_{☾}^2 r^2}{\epsilon} \left\{\frac{\partial \cos S'_{10}}{\cos i_{☾} \, \partial i_{☾}}\right\}_{i_{☾}=0} \sin i_{☾} \cos \tilde{S}_{10} .$$

From the expression for the $\cos S'_{10}$ it is easily found that the quantity in the bracket is

$$\sin I \sin v_{\oplus} \sin (f + g).$$

Thus

$$F_3(i_{☾}) = 3 \frac{n_{☾}^2 r^2}{\epsilon} \cos \tilde{S}'_{10} \sin (f + g) \sin v_{\oplus} \sin I \sin i_{☾} .$$

Now

$$\cos \tilde{S}'_{10} \sin (f + g) = \sin (f + g) \cos (f + g) \cos h$$

$$- \sin^2 (f + g) \sin h \cos I .$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \sin(f+g)\cos(f+g)\, dl = \frac{5}{4}\, a^2 e^2 \sin 2g$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \sin^2(f+g)\, dl = \frac{1}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi} r^2\, dl\right.$$

$$\left. - \frac{1}{2\pi}\int_0^{2\pi} r^2 \cos 2(f+g)\, dl\right\} =$$

$$= \frac{1}{2}\left\{a^2\left(1+\frac{3}{2}e^2\right) - \frac{5}{2}\, a^2 e^2 \cos 2g\right\},$$

the part of $F_3(i_☾)$ which is free from short periodic terms is

$$F_3'(i_☾) = \frac{3n_☾^2 a'^2}{8\epsilon} \sin v_\oplus \sin I' \sin i_☾ \,\{-2(2+3e'^2)\cos I' \sin h'$$

$$+ 5\, e'^2(1+\cos I')\sin(2g'+h') + 5\, e'^2(1-\cos I')\sin(2g'-h')\}.$$

There is no contribution to F_3'' and S_2'. The contribution to S_3' is

$$S_3'(i_☾) = -\frac{3n_☾^2\, a''^2}{8\epsilon(n_☾ + N\omega_☾)} \cos v_\oplus \sin I' \sin i_☾ \,\{-2(2+3e''^2)\cos I'' \sin h''$$

$$+ 5e''^2(1+\cos I'')\sin(2g''+h'') + 5e''^2(1-\cos I'')\sin(2g''-h'')\},$$

where $N\omega_☾$ is given by $\omega_☾ = N\omega_☾\, t + \text{const}$. Again, this is a second order contribution since $n_☾^2/(n_☾ + N\omega_☾) \simeq n_☾$.

20. The Non-Sphericity of the Potential Field of the Earth

Because of the fact that the earth's equator is not the reference plane, the form of the disturbing function due to the zonal harmonic coefficient j_2 of the earth will be derived from basic relations.

In Figure 5, E is the earth, M the moon and S the orbiter.

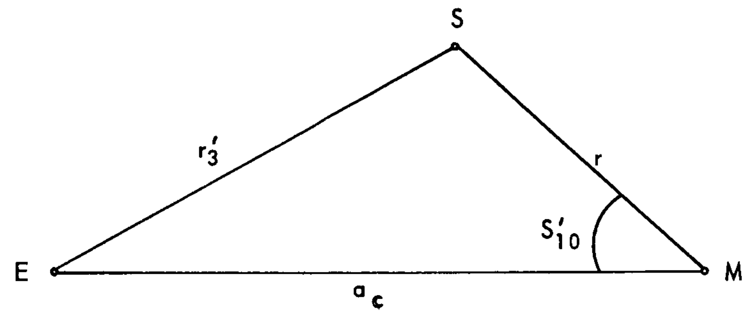

Figure 5

The disturbing function for the motion of S is given by

$$F_{\oplus} = \frac{k^2 m_{\oplus}}{r_3'} \left\{ 1 - \frac{R_{\oplus}^2 j_2}{r_3'^2} P_2(\sin \varphi) \right\},$$

where $R_{\oplus}$ is the equatorial radius of the earth and φ the latitude of the orbiter with respect to the earth's equator. The part $k^2 m_{\oplus}/r'$ has already been taken into account and the rest is supposed to be a 3rd order quantity. Therefore

$$F_3(\oplus) = - \frac{k^2 m_{\oplus}}{r_3'^3} R_{\oplus}^2 j_2 P_2 (\sin \varphi).$$

If the terms $(a/a_{☾}) j_2$ and $e_{☾} j_2$ are neglected, it follows that

$$F_3(\oplus) = - \frac{n_{☾}^2 R_{\oplus}^2 j_2}{\epsilon} P_2 (\sin \varphi),$$

assuming $a_{☾}/r_3' = 1$.

The angle φ must now be expressed in terms of the orbital elements of the lunar equator.

In Figure 6 the geometry of the problem is given.

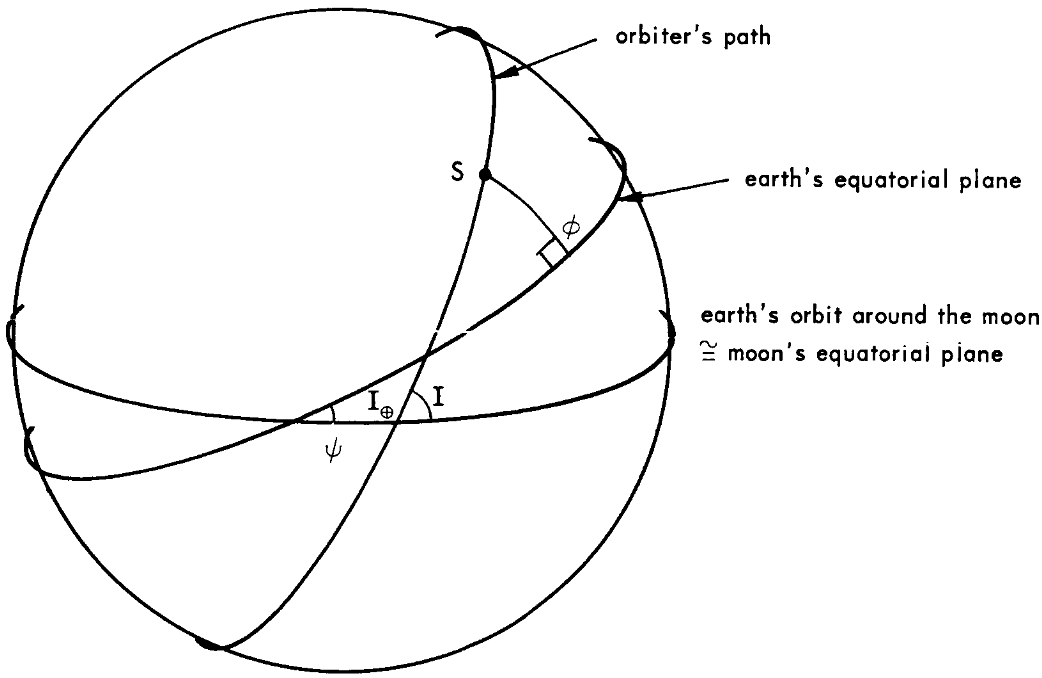

Figure 6

From basic relations of spherical trigonometry it is found that

$$\sin \varphi = (\sin I \cos I_{\oplus} - \cos \psi \cos I \sin I_{\oplus}) \sin (f + g)$$

$$- \sin I_{\oplus} \sin \psi \cos (f + g).$$

Then, if

$$\bar{A} = \sin I \cos I_{\oplus} - \cos \psi \cos I \sin I_{\oplus}$$

$$\bar{B} = - \sin I_{\oplus} \sin \psi,$$

we have

$$F_3(\oplus) = -\frac{n_{\mathbb{C}}^2 R_{\oplus}^2 j_2}{\epsilon}\left\{\left[\frac{3}{4}(\overline{A}^2+\overline{B}^2)-\frac{1}{2}\right]+\frac{3}{4}(\overline{B}^2-\overline{A}^2)\cos 2(f+g)\right.$$

$$\left.+\frac{3}{2}\overline{A}\overline{B}\sin 2(f+g)\right\}.$$

In order to eliminate the short periodic terms, the following integrals are needed:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos 2f\,dl = 2\beta^2\left(\frac{1}{2}+\sqrt{1-e^2}\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin 2f\,dl = 0,$$

where

$$\beta = \frac{1}{e}\left(1-\sqrt{1-e^2}\right).$$

(The use of β here is not to be confused with its use earlier as the latitude.) They are easily obtained by considering the expansion

$$dl = df + 2\sum_{k=1}^{\infty} k\beta^k\left(\frac{1}{k}+\sqrt{1-e^2}\right)\cos kf\,(-1)^k\,df,$$

which is the differential of the equation of the center. It follows that

$$F_3'(\oplus) = -\frac{n_{\mathbb{C}}^2 R_{\oplus}^2 j_2}{\epsilon}\left\{\frac{3}{4}(\overline{A}^2+\overline{B}^2)-\frac{1}{2}+\frac{3}{2}\beta'^2\left(\frac{1}{2}+\sqrt{1-e'^2}\right)\cdot\right.$$

$$\left.\cdot\left[(\overline{B}^2-\overline{A}^2)\cos 2g' + 2\overline{A}\,\overline{B}\sin 2g'\right]\right\},$$

where

$$\beta' = \beta(e').$$

Furthermore, since $\psi = h + \lambda_\oplus - \bar{\Omega}'$, where $\bar{\Omega}'$ is the longitude of the descending node of the moon's equator on the earth's equator, we have

$$\bar{B}^2 + \bar{A}^2 = \left(\sin^2 I' \cos^2 I_\oplus + \frac{1}{2}\cos^2 I' \sin^2 I_\oplus + \frac{1}{2}\sin^2 I_\oplus\right)$$

$$-\frac{1}{2}\sin^2 I_\oplus \sin^2 I' \cos 2(h' + \lambda_\oplus - \bar{\Omega}')$$

$$-\frac{1}{2}\sin 2I_\oplus \sin 2I' \cos (h' + \lambda_\oplus - \bar{\Omega}')$$

$$\bar{B}^2 - \bar{A}^2 = \left(\frac{1}{2}\sin^2 I_\oplus - \sin^2 I' \cos^2 I_\oplus - \frac{1}{2}\cos^2 I' \sin^2 I_\oplus\right)$$

$$-\frac{1}{2}\sin^2 I_\oplus (1 + \cos^2 I') \cos 2(h' + \lambda_\oplus - \bar{\Omega}')$$

$$+\frac{1}{2}\sin 2I' \sin 2I_\oplus \cos (h' + \lambda_\oplus - \bar{\Omega}'),$$

and

$$\bar{A}\bar{B} = \frac{1}{2}\sin^2 I_\oplus \cos I' \sin 2(h' + \lambda_\oplus - \bar{\Omega}') - \frac{1}{2}\sin 2I_\oplus \sin I' \sin (h' + \lambda_\oplus - \bar{\Omega}').$$

Thus, the contribution to S_2' is zero, and the contribution to F_3'' is

$$F_3''(\oplus) = -\frac{1}{8}\frac{n_☾^2 j_2 R_\oplus^2}{\epsilon}\left\{-(1 - 3\cos^2 I'')\right.$$

$$\left. + 3\beta''^2\left(1 + 2\sqrt{1 - e''^2}\right)\sin^2 I'' \cos 2g''\right\}(1 - 3\cos^2 I_\oplus).$$

The contribution to S_3' will be given by the integration of

$$\frac{\partial S_3'(\oplus)}{\partial \tau} = -\frac{3}{16}\,\frac{n_{\mathbb{C}}^2\, j_2\, R_\oplus^2 \sin I_\oplus}{\epsilon}\Big\{-2\sin I''\,[\sin I'' \sin I_\oplus \cos 2(h' + \lambda_\oplus - \overline{\Omega}')$$

$$+ 4\cos I'' \cos I_\oplus \cos(h' + \lambda_\oplus - \overline{\Omega}')]$$

$$+ \beta''^2\left(1 + 2\sqrt{1 - e''^2}\right)[-(1 + \cos I'')^2 \sin I_\oplus \cos 2(h' + g' + \lambda_\oplus - \overline{\Omega}')$$

$$- (1 - \cos I'')^2 \sin I_\oplus \cos 2(h' - g' + \lambda_\oplus - \overline{\Omega}')$$

$$+ 4\sin I''(1 + \cos I'')\cos I_\oplus \cos(h' + 2g' + \lambda_\oplus - \overline{\Omega}')$$

$$- 4\sin I''(1 - \cos I'')\cos I_\oplus \cos(h' - 2g' + \lambda_\oplus - \overline{\Omega}')]\Big\}.$$

Since

$$\frac{\partial}{\partial \tau}(\lambda_\oplus - \overline{\Omega}') = n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}},$$

we have

$$S_3'(\oplus) = -\frac{3}{32}\,\frac{n_{\mathbb{C}}^2\, j_2\, R_\oplus^2 \sin I_\oplus}{\epsilon(n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}})}\Big\{-2\sin I''\,[\sin I'' \sin I_\oplus \sin 2(h'' + \lambda_\oplus - \overline{\Omega}')$$

$$+ 8\cos I'' \cos I_\oplus \sin(h'' + \lambda_\oplus - \overline{\Omega}')]$$

$$+ \beta''^2\left(1 + 2\sqrt{1 - e''^2}\right)[-(1 + \cos I'')^2 \sin I_\oplus \sin 2(h'' + g'' + \lambda_\oplus - \overline{\Omega}')$$

$$-(1 - \cos I'')^2 \sin I_\oplus \sin 2(h'' - g'' + \lambda_\oplus - \overline{\Omega}')$$

$$+8\sin I''(1 + \cos I'')\cos I_\oplus \sin(h'' + 2g'' + \lambda_\oplus - \overline{\Omega}')$$

$$-8\sin I''(1 - \cos I'')\cos I_\oplus \sin(h'' - 2g'' + \lambda_\oplus - \overline{\Omega}')]\Big\}.$$

21. Higher Order Zonal Harmonics for the Moon's Potential Field

The terms corresponding to J_3, J_4, J_5 are obtained immediately from Brouwer's theory for artificial satellites (Reference 1). Their contribution is entirely to the "secular" part of F_3'. So

$$F_3''(J_{3,4,5}) = -\frac{3}{8}\,\frac{n'^2 J_3 b^3}{a'(1-e''^2)^{5/2}}\; e'' \sin I''(1 - 5\cos^2 I'')\sin g''$$

$$-\frac{3}{128}\,\frac{n'^2 J_4 b^4}{a'^2(1-e''^2)^{7/2}}\;[(3 - 30\cos^2 I'' + 35\cos^4 I'')(2 + 3e''^2)$$

$$- 10(1 - 8\cos^2 I'' + 7\cos^4 I'')e''^2 \cos 2g'']$$

$$-\frac{5}{256}\,\frac{n'^2 J_5 b^5}{a'^3(1-e''^2)^{9/2}}\; e'' \sin I''\,[6(1 - 14\cos^2 I'' + 21\cos^4 I'')\cdot$$

$$\cdot (4 + 3e''^2)\sin g'' - 7(1 - 10\cos^2 I'' + 9\cos^4 I'')\,e''^2 \sin 3g''] .$$

The orders of magnitude of these terms will depend, for reasonable values of e'', on the values of J_3, J_4, and J_5, which are not very well known.

22. Physical Libration, and the Precession of the Lunar Equator

Physical libration causes a periodic oscillation in the position of the lunar surface, thereby creating a small angular displacement between the principal lunar meridian and the earth-moon line of centers. The largest contribution to this displacement is given by

$$\chi = \alpha \sin \ell_\odot + \chi_0 \quad \text{(see Reference 9, p. 316)}$$

where

$$\alpha = -59'' = -2.86 \times 10^{-4} \text{ rad.}$$

χ_0 = constant dependent upon the initial time

$\ell_{\odot}$ = mean anomaly of the sun.

Let $\xi\ \eta\ \zeta$ and $x\,y\,z$ be lunicentric, equatorial coordinate systems with the ξ-axis directed toward the earth and the x-axis passing through the principal meridian (see Figure 7).

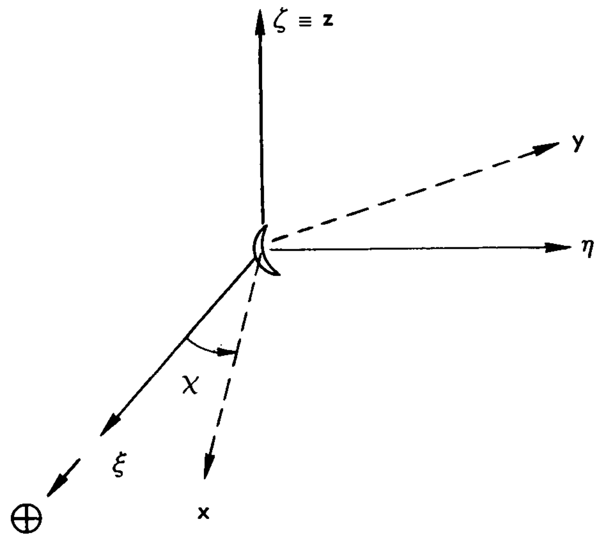

Figure 7

Then,

$$\ddot{\xi} = \frac{\partial F}{\partial \xi}, \quad \ddot{\eta} = \frac{\partial F}{\partial \eta}$$

$$x = \xi \cos \chi + \eta \sin \chi$$

$$y = -\xi \sin \chi + \eta \cos \chi.$$

Thus,

$$\ddot{x} = \ddot{\xi} \cos \chi + \ddot{\eta} \sin \chi + 2\alpha n_{\odot} \cos \ell_{\odot} (\dot{\eta} \cos \chi - \dot{\xi} \sin \chi)$$
$$- \alpha n_{\odot}^2 \sin \ell_{\odot} (\eta \cos \chi - \xi \sin \chi) - \alpha^2 n_{\odot}^2 \cos^2 \ell_{\odot} (\eta \sin \chi + \xi \cos \chi),$$

where $n_\odot$ = mean motion in the mean anomaly of the sun.

But

$$\dot{\eta} \cos \chi - \dot{\xi} \sin \chi = \alpha n_\odot x \cos \ell_\odot + \dot{y}$$

$$\ddot{\xi} \cos \chi + \ddot{\eta} \sin \chi = \frac{\partial F}{\partial x} ,$$

so

$$\ddot{x} = \frac{\partial F}{\partial x} + 2\alpha n_\odot \cos \ell_\odot (\alpha n_\odot x \cos \ell_\odot + \dot{y}) - \alpha n_\odot^2 \sin \ell_\odot (\eta \cos \chi - \xi \sin \chi)$$
$$- \alpha^2 n_\odot^2 x \cos^2 \ell_\odot .$$

Now

$$\frac{\alpha^2 n_\odot^2}{n^2} \sim O(10^{-12})$$

$$\left| \frac{\alpha n_\odot^2}{n^2} \right| \sim O(10^{-9})$$

$$\left| \frac{\alpha n_\odot}{n} \right| \sim O(10^{-6}).$$

Therefore, if one neglects terms higher than third order, one has

$$\ddot{x} = \frac{\partial F}{\partial x} + 2\alpha n_\odot \dot{y} \cos \ell_\odot .$$

A similar computation yields

$$\ddot{y} = \frac{\partial F}{\partial y} - 2 \alpha n_\odot \dot{x} \cos \ell_\odot .$$

Thus,

$$\ddot{x} = \frac{\partial \bar{F}}{\partial x}, \; \ddot{y} = \frac{\partial \bar{F}}{\partial y} ,$$

where $$\overline{F} = F + 2\alpha n_\odot (x\dot{y} - \dot{x}y) \cos \ell_\odot$$

$$= F + 2\alpha n_\odot H \cos \ell_\odot .$$

Hence, the addition to the Hamiltonian of the problem is

$$F_3(\text{lib.}) = 2\alpha n_\odot H \cos \ell_\odot .$$

Then,

$$F_{3_s}(\text{lib.}) \equiv 0$$

$$F_{3_{ph'}}(\text{lib.}) \equiv 0$$

$$F_{3_{p\tau}}(\text{lib.}) = 2\alpha n_\odot H \cos \ell_\odot ,$$

and therefore

$$\frac{\partial S_3'(\text{lib.})}{\partial \tau} = 2\alpha n_\odot H \cos \ell_\odot \Longrightarrow S_3'(\text{lib.}) = 2\alpha H \sin \ell_\odot .$$

Now, the precession in the lunar equator is created by the small angle of inclination ($\sim 1^\circ 32'.1$) between the lunar equatorial plane and the ecliptic. This results in the regression of the equatorial node in the ecliptic. However, the motion of the node is already implicit in the $n_☾^*$ – hence, no additional corrections need be made.

23. Complete "Secular" Third Order Hamiltonian

The complete secular third order Hamiltonian is given by

$$F_3'' = F_3'' \text{ (coupling)} + F_3'' \text{ (sun)} + F_3'' (\oplus) + F_3'' (J_{3,4,5}) .$$

Formulas for the various parts of F_3'' can be found in preceding sections.

24. The Determining Function for Long Period Terms

Additional long periodic perturbations depending on the motion of the node, the motion of the earth and the motion of the sun will be given through the determining functions S_2' and S_3'. The determining functions S_2' and S_3' are

$$S_2' = S_2' \text{ (coupling)} + S_2' \text{ (earth)}$$

$$S_3' = S_3' \text{ (radiation)} + S_3' \text{(sun)} + S_3' (e_{☾})$$

$$+ S_3' (i_{☾}) + S_3' (\oplus) + S_3' \text{ (libration)}.$$

In the next section, the partial derivatives needed in order to find additional perturbations in the canonical elements are given.

25. Long Period Perturbations of Second Order

The perturbations of long-period are obtained from

$$l'' = \frac{\partial S}{\partial L''} = l' + \frac{\partial S_1'}{\partial L''} + \frac{\partial S_2'}{\partial L''} + \frac{\partial S_3'}{\partial L''}, \qquad L' = \frac{\partial S}{\partial l'} = L''$$

$$g'' = \frac{\partial S}{\partial G''} = g' + \frac{\partial S_1'}{\partial G''} + \frac{\partial S_2'}{\partial G''} + \frac{\partial S_3'}{\partial G''}, \qquad G' = \frac{\partial S}{\partial g'} = G'' + \frac{\partial S_1'}{\partial g'} + \frac{\partial S_2'}{\partial g'} + \frac{\partial S_3'}{\partial g'}$$

$$h'' = \frac{\partial S}{\partial H''} = h' + \frac{\partial S_1'}{\partial H''} + \frac{\partial S_2'}{\partial H''} + \frac{\partial S_3'}{\partial H''}, \qquad H' = \frac{\partial S}{\partial h'} = H'' + \frac{\partial S_1'}{\partial h'} + \frac{\partial S_2'}{\partial h'} + \frac{\partial S_3'}{\partial h'}$$

The terms corresponding to S_1' have already been obtained. Next the partial derivatives of S_2' and S_3' with respect to a'', e'', I'', g', and h' are computed. For convenience, however, the primes have been dropped in this section.

The "mean" mean motion n' (called n) depends on a' (called a) through the relation

$$n^2 = \frac{\mu}{a^3} ,$$

and

$$\frac{\partial S}{\partial L} = 2\sqrt{\frac{a}{\mu}} \; \frac{\partial S}{\partial a} + \frac{1-e^2}{e\sqrt{\mu a}} \; \frac{\partial S}{\partial e}$$

$$\frac{\partial S}{\partial G} = -\frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}} \; \frac{\partial S}{\partial e} + \frac{\cot I}{\sqrt{\mu a\,(1-e^2)}} \; \frac{\partial S}{\partial I}$$

$$\frac{\partial S}{\partial H} = -\frac{1}{\sin I\sqrt{\mu a\,(1-e^2)}} \; \frac{\partial S}{\partial I} .$$

The partial derivatives of the various parts of S_2' and S_3' are as follows:

$$\begin{aligned}
\frac{\partial S_2' \text{ (coupling)}}{\partial a} = &+ \frac{63}{8192\,\epsilon^2}\left(\frac{n_{☾}}{n}\right)^2\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 \frac{na}{\sqrt{1-e^2}} \{ 32\sin^2 I\cos I\,(1-e^2)(2-17e^2)\sin 2h \\
&- 4\sin^2 I\cos I\,(2+3e^2)^2 \sin 4h \\
&+ 80\,(1+\cos I)^2\,(2-3\cos I)\,e^2\,(1-e^2)\sin(2g+2h) \\
&+ 80\,(1-\cos I)^2\,(2+3\cos I)\,e^2\,(1-e^2)\sin(2g-2h) \\
&+ 10\sin^2 I\,(1+\cos I)\,e^2\,[6e^2-(1+5\cos I)]\sin(2g+4h) \\
&+ 10\sin^2 I\,(1-\cos I)\,e^2\,[6e^2-(1-5\cos I)]\sin(2g-4h)
\end{aligned}$$

$$+25(1+\cos I)^3 e^2 [2e^2-(1+\cos I)] \sin(4g+4h)$$

$$+25(1-\cos I)^3 e^2 [2e^2-(1-\cos I)] \sin(4g-4h) \}$$

$$-\frac{27}{512\,\epsilon}\left(\frac{n_{☾}}{n^*_{☾}}\right)^2 J_2 b^2 \frac{n}{a(1-e^2)^2} \{4\sin^2 I\cos I(2+3e^2)\sin 2h$$

$$+5(1+\cos I)^2(1+2\cos I-5\cos^2 I)e^2\sin(2g+2h)$$

$$+5(1-\cos I)^2(1-2\cos I-5\cos^2 I)e^2\sin(2g-2h)\}$$

$$+\frac{27}{1024\,\epsilon}\left(\frac{n_{☾}}{n^*_{☾}}\right)^2 J_{22} b^2 \frac{n\sin^2 I}{a(1-e^2)^2} \{-8\cos I(2+3e^2)(2\sin 2h-\sin 4h)$$

$$-20(1+\cos I)(3-5\cos I)e^2\sin(2g+2h)$$

$$-20(1-\cos I)(3+5\cos I)e^2\sin(2g-2h)$$

$$+5(1+\cos I)(1+5\cos I)e^2\sin(2g+4h)$$

$$+5(1-\cos I)(1-5\cos I)e^2\sin(2g-4h)\}$$

$$-\frac{117}{16}\left(\frac{n}{n^*_{☾}}\right)^2 J_2 J_{22} b^4 \frac{n\sin^2 I\cos I}{a^3(1-e^2)^{7/2}}\sin 2h$$

$$+\frac{117}{32}\left(\frac{n}{n^*_{☾}}\right)^2 J_{22}^2 b^4 \frac{n\sin^2 I\cos I}{a^3(1-e^2)^{7/2}}\sin 4h$$

$$\frac{\partial S_2' \text{ (coupling)}}{\partial I} = + \frac{9}{2048\,\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n}\right)^2 \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 \frac{n\,a^2 \sin I}{\sqrt{1-e^2}} \cdot$$

$$\cdot \{ - 16(1-3\cos^2 I)(1-e^2)(2-17e^2) \sin 2h$$

$$+ 2(1-3\cos^2 I)(2+3e^2)^2 \sin 4h$$

$$- 40(1+\cos I)(1-9\cos I)\, e^2 (1-e^2) \sin(2g+2h)$$

$$+ 40(1-\cos I)(1+9\cos I)\, e^2 (1-e^2) \sin(2g-2h)$$

$$+ 10(1+\cos I)\, e^2 \,[(3-5\cos I)(1+2\cos I)$$

$$-3e^2(1-3\cos I)] \sin(2g+4h)$$

$$- 10(1-\cos I)\, e^2 \,[(3+5\cos I)(1-2\cos I)$$

$$- 3e^2(1+3\cos I)] \sin(2g-4h)$$

$$+ 25(1+\cos I)^2 e^2 \,[2(1+\cos I) - 3e^2] \sin(4g+4h)$$

$$- 25(1-\cos I)^2 e^2 \,[2(1-\cos I) - 3e^2] \sin(4g-4h) \}$$

$$+ \frac{9}{64\,\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_2\, b^2 \frac{n \sin I}{(1-e^2)^2} \{ -(1-3\cos^2 I)(2+3e^2) \sin 2h$$

$$- 5(1+\cos I)(1-\cos I - 5\cos^2 I)\, e^2 \sin(2g+2h)$$

$$+ 5(1-\cos I)(1+\cos I - 5\cos^2 I)\, e^2 \sin(2g-2h) \}$$

$$- \frac{9}{256\,\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_{22}\, b^2\, \frac{n \sin I}{(1-e^2)^2} \{ 4(1-3\cos^2 I)(2+3e^2)(2\sin 2h - \sin 4h)$$

$$- 20(1+\cos I)(1+7\cos I - 10\cos^2 I)\, e^2 \sin(2g+2h)$$

$$+ 20(1-\cos I)(1-7\cos I - 10\cos^2 I)\, e^2 \sin(2g-2h)$$

$$- 5(1+\cos I)(1+2\cos I)(3-5\cos I)\, e^2 \sin(2g+4h)$$

$$+ 5(1-\cos I)(1-2\cos I)(3+5\cos I)\, e^2 \sin(2g-4h)\}$$

$$- \frac{9}{8}\left(\frac{n}{n^*_{\mathbb{C}}}\right)^2 J_2 J_{22}\, b^4\, \frac{n \sin I\,(1-3\cos^2 I)}{a^2\,(1-e^2)^{7/2}} \sin 2h$$

$$+ \frac{9}{16}\left(\frac{n}{n^*_{\mathbb{C}}}\right)^2 J_{22}^2\, b^4\, \frac{n \sin I\,(1-3\cos^2 I)}{a^2\,(1-e^2)^{7/2}} \sin 4h$$

$$\frac{\partial S_2' \text{ (coupling)}}{\partial e} = + \frac{9}{4096\,\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n}\right)^2 \left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 \frac{n\, a^2\, e}{(1-e^2)^{3/2}} \cdot$$

$$\cdot\{-96\sin^2 I \cos I\,(12-17e^2)(1-e^2)\sin 2h$$

$$-4\sin^2 I\cos I\,(2+3e^2)(14-9e^2)\sin 4h$$

$$+80(1+\cos I)^2(2-3\cos I)(2-3e^2)(1-e^2)\sin(2g+2h)$$

$$+80(1-\cos I)^2(2+3\cos I)(2-3e^2)(1-e^2)\sin(2g-2h)$$

$$+10\sin^2 I(1+\cos I)\,[6e^2(4-3e^2)-(2-e^2)(1+5\cos I)]\sin(2g+4h)$$

$$+10\sin^2 I(1-\cos I)\,[6e^2(4-3e^2)-(2-e^2)(1-5\cos I)]\sin(2g-4h)$$

$$+25(1+\cos I)^3\,[2e^2(4-3e^2)-(2-e^2)(1+\cos I)]\sin(4g+4h)$$

$$+25(1-\cos I)^3\,[2e^2(4-3e^2)-(2-e^2)(1-\cos I)]\sin(4g-4h)\}$$

$$+\frac{9}{128\epsilon}\left(\frac{n_☾}{n_☾^*}\right)^2 J_2 b^2 \frac{ne}{(1-e^2)^3} \{4\sin^2 I\cos I\,(7+3e^2)\sin 2h$$

$$+5(1+\cos I)^2(1+2\cos I-5\cos^2 I)(1+e^2)\sin(2g+2h)$$

$$+5(1-\cos I)^2(1-2\cos I-5\cos^2 I)(1+e^2)\sin(2g-2h)\}$$

$$-\frac{9}{256\epsilon}\left(\frac{n_☾}{n_☾^*}\right)^2 J_{22} b^2 \frac{ne\sin^2 I}{(1-e^2)^3} \{-8\cos I\,(7+3e^2)(2\sin 2h-\sin 4h)$$

$$-20(1+\cos I)(3-5\cos I)(1+e^2)\sin(2g+2h)$$

$$-20(1-\cos I)(3+5\cos I)(1+e^2)\sin(2g-2h)$$

$$+5(1+\cos I)(1+5\cos I)(1+e^2)\sin(2g+4h)$$

$$+5(1-\cos I)(1-5\cos I)(1+e^2)\sin(2g-4h)\}$$

$$+\frac{63}{8}\left(\frac{n}{n_☾^*}\right)^2 J_2 J_{22} b^4 \frac{ne\sin^2 I\cos I}{a^2(1-e^2)^{9/2}}\sin 2h$$

$$-\frac{63}{16}\left(\frac{n}{n_☾^*}\right)^2 J_{22}^2 b^4 \frac{ne\sin^2 I\cos I}{a^2(1-e^2)^{9/2}}\sin 4h$$

$$\frac{\partial S_2'\text{ (coupling)}}{\partial g} = +\frac{45}{1024\epsilon^2}\left(\frac{n_☾}{n}\right)^2\left(\frac{n_☾}{n_☾^*}\right)^2 \frac{n\,a^2 e^2}{\sqrt{1-e^2}} \{8(1+\cos I)^2(2-3\cos I)(1-e^2)\cdot$$

$$\cdot\cos(2g+2h)$$

$$+8(1-\cos I)^2(2+3\cos I)(1-e^2)\cos(2g-2h)$$

$$+\sin^2 I\,(1+\cos I)\,[6e^2-(1+5\cos I)]\cos(2g+4h)$$

$$+\sin^2 I\,(1-\cos I)\,[6e^2-(1-5\cos I)]\cos(2g-4h)$$

$$+5(1+\cos I)^3\ [2e^2-(1+\cos I)]\cos(4g+4h)$$

$$+5(1-\cos I)^3\ [2e^2-(1-\cos I)]\cos(4g-4h)\ \}$$

$$+\frac{45}{128\,\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_2\, b^2\, \frac{ne^2}{(1-e^2)^2}\ \{(1+\cos I)^2(1+2\cos I-5\cos^2 I)\cos(2g+2h)$$

$$+(1-\cos I)^2(1-2\cos I-5\cos^2 I)\cos(2g-2h)\}$$

$$-\frac{45}{256\,\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_{22}\, b^2\, \frac{ne^2\sin^2 I}{(1-e^2)^2}\ \{-4(1+\cos I)(3-5\cos I)\cos(2g+2h)$$

$$-4(1-\cos I)(3+5\cos I)\cos(2g-2h)$$

$$+(1+\cos I)(1+5\cos I)\cos(2g+4h)$$

$$+(1-\cos I)(1-5\cos I)\cos(2g-4h)\ \}$$

$$\frac{\partial S_2'\ (\text{coupling})}{\partial h} = +\frac{9}{1024\,\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n}\right)^2\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2\frac{na^2}{\sqrt{1-e^2}}\ \{16\sin^2 I\cos I\,(1-e^2)(2-17e^2)\cos 2h$$

$$-4\sin^2 I\cos I\,(2+3e^2)^2\cos 4h$$

$$+40(1+\cos I)^2(2-3\cos I)\,e^2(1-e^2)\cos(2g+2h)$$

$$-40(1-\cos I)^2(2+3\cos I)\,e^2(1-e^2)\cos(2g-2h)$$

$$+10\sin^2 I\,(1+\cos I)\,e^2[6e^2-(1+5\cos I)]\cos(2g+4h)$$

$$-10\sin^2 I\,(1-\cos I)\,e^2[6e^2-(1-5\cos I)]\cos(2g-4h)$$

$$+25(1+\cos I)^3 e^2\ [2e^2-(1+\cos I)]\cos(4g+4h)$$

$$-25(1-\cos I)^3 e^2\ [2e^2-(1-\cos I)]\cos(4g-4h)\}$$

$$+\frac{9}{128\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_2\, b^2\,\frac{n}{(1-e^2)^2}\ \{4\sin^2 I\cos I\,(2+3e^2)\cos 2h$$

$$+5(1+\cos I)^2(1+2\cos I-5\cos^2 I)\,e^2\cos(2g+2h)$$

$$-5(1-\cos I)^2(1-2\cos I-5\cos^2 I)\,e^2\cos(2g-2h)\}$$

$$-\frac{9}{128\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_{22}\, b^2\,\frac{n\sin^2 I}{(1-e^2)^2}\ \{-8\cos I\,(2+3e^2)(\cos 2h-\cos 4h)$$

$$-10\,(1+\cos I)(3-5\cos I)\,e^2\cos(2g+2h)$$

$$+10\,(1-\cos I)(3+5\cos I)\,e^2\cos(2g-2h)$$

$$+5\,(1+\cos I)(1+5\cos I)\,e^2\cos(2g+4h)$$

$$-5\,(1-\cos I)(1-5\cos I)\,e^2\cos(2g-4h)\}$$

$$+\frac{9}{4}\left(\frac{n}{n^*_{\mathbb{C}}}\right)^2 J_2\, J_{22}\, b^4\,\frac{n\sin^2 I\cos I}{a^2\,(1-e^2)^{7/2}}\cos 2h$$

$$-\frac{9}{4}\left(\frac{n}{n^*_{\mathbb{C}}}\right)^2 J_{22}^2\, b^4\,\frac{n\sin^2 I\cos I}{a^2\,(1-e^2)^{7/2}}\cos 4h$$

$$\frac{\partial S_2'\,(\mathrm{earth})}{\partial a}=\frac{5}{512\,\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)\frac{n_{\mathbb{C}}\,a^2\,e}{a_{\mathbb{C}}}\ \{-9(1+\cos I)(1+10\cos I-15\cos^2 I)(4+3e^2)\sin(g+h)$$

$$+9(1-\cos I)(1-10\cos I-15\cos^2 I)(4+3e^2)\sin(g-h)$$

$$+15\sin^2 I\,(1+\cos I)(4+3e^2)\sin(g+3h)$$

$$-15\sin^2 I\,(1-\cos I)(4+3e^2)\sin(g-3h)$$

$$+315\sin^2 I\,(1+\cos I)\,e^2\sin(3g+h)$$

$$-315\sin^2 I\,(1-\cos I)\,e^2\sin(3g-h)$$

$$+35(1+\cos I)^3 e^2\sin(3g+3h)$$

$$-35(1-\cos I)^3 e^2\sin(3g-3h)\}$$

$$\frac{\partial S_2'\,(\text{earth})}{\partial I} = \frac{5}{512\,\epsilon}\left(\frac{n_{\text{c}}}{n_{\text{c}}^*}\right)\frac{n_{\text{c}}\,a^3 e\sin I}{a_{\text{c}}}\{3(11-10\cos I-45\cos^2 I)(4+3e^2)\sin(g+h)$$

$$+3(11+10\cos I-45\cos^2 I)(4+3e^2)\sin(g-h)$$

$$-5(1+\cos I)(1-3\cos I)(4+3e^2)\sin(g+3h)$$

$$-5(1-\cos I)(1+3\cos I)(4+3e^2)\sin(g-3h)$$

$$-105(1+\cos I)(1-3\cos I)\,e^2\sin(3g+h)$$

$$-105(1-\cos I)(1+3\cos I)\,e^2\sin(3g-h)$$

$$-35(1+\cos I)^2 e^2\sin(3g+3h)$$

$$-35(1-\cos I)^2 e^2\sin(3g-3h)\}$$

$$\frac{\partial S_2'\,(\text{earth})}{\partial e} = \frac{5}{512\,\epsilon}\left(\frac{n_{\text{c}}}{n_{\text{c}}^*}\right)\frac{n_{\text{c}}\,a^3}{a_{\text{c}}}\{-3(1+\cos I)(1+10\cos I-15\cos^2 I)(4+9e^2)\sin(g+h)$$

$$+3(1-\cos I)(1-10\cos I-15\cos^2 I)(4+9e^2)\sin(g-h)$$

$$+5\sin^2 I\,(1+\cos I)(4+9e^2)\sin(g+3h)$$

$$- 5 \sin^2 I\,(1-\cos I)(4+9e^2) \sin (g-3h)$$

$$+ 315 \sin^2 I\,(1+\cos I)\, e^2 \sin (3g+h)$$

$$- 315 \sin^2 I\,(1-\cos I)\, e^2 \sin (3g-h)$$

$$+ 35\,(1+\cos I)^3\, e^2 \sin (3g+3h)$$

$$- 35\,(1-\cos I)^3\, e^2 \sin (3g-3h) \}$$

$$\frac{\partial S_2' \text{ (earth)}}{\partial g} = \frac{5}{512\,\epsilon} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right) \frac{n_{\mathbb{C}}\, a^3\, e}{a_{\mathbb{C}}} \{- 3\,(1+\cos I)(1+10\cos I-15\cos^2 I)(4+3e^2) \cos (g+h)$$

$$+ 3\,(1-\cos I)(1-10\cos I-15\cos^2 I)(4+3e^2) \cos (g-h)$$

$$+ 5 \sin^2 I\,(1+\cos I)(4+3e^2) \cos (g+3h)$$

$$- 5 \sin^2 I\,(1-\cos I)(4+3e^2) \cos (g-3h)$$

$$+ 315 \sin^2 I\,(1+\cos I)\, e^2 \cos (3g+h)$$

$$- 315 \sin^2 I\,(1-\cos I)\, e^2 \cos (3g-h)$$

$$+ 35\,(1+\cos I)^3\, e^2 \cos (3g+3h)$$

$$- 35\,(1-\cos I)^3\, e^2 \cos (3g-3h) \}$$

$$\frac{\partial S_2' \text{ (earth)}}{\partial h} = \frac{5}{512\,\epsilon} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right) \frac{n_{\mathbb{C}}\, a^3\, e}{a_{\mathbb{C}}} \{-3(1+\cos I)(1+10\cos I-15\cos^2 I)(4+3e^2) \cos (g+h)$$

$$- 3\,(1-\cos I)(1-10\cos I-15\cos^2 I)(4+3e^2) \cos (g-h)$$

$$+ 15 \sin^2 I\,(1+\cos I)(4+3e^2) \cos (g+3h)$$

$$+ 15 \sin^2 I (1-\cos I)(4+3e^2) \cos (g-3h)$$

$$+ 105 \sin^2 I (1+\cos I)\, e^2 \cos (3g+h)$$

$$+ 105 \sin^2 I (1-\cos I)\, e^2 \cos (3g-h)$$

$$+ 35 (1+\cos I)^3\, e^2 \cos (3g+3h)$$

$$+ 35 (1-\cos I)^3\, e^2 \cos (3g-3h) \}$$

$$\frac{\partial S_3' \text{ (radiation)}}{\partial a} = \frac{3}{2} \frac{\sigma e}{n_{☾}^*-n_{\oplus}^*} \left\{ \cos^2 \frac{I}{2} \sin (h+g+\lambda_{\oplus}-\lambda_{\odot}) + \sin^2 \frac{I}{2} \sin (h-g+\lambda_{\oplus}-\lambda_{\odot}) \right\}$$

$$\frac{\partial S_3' \text{ (radiation)}}{\partial I} = \frac{3}{4} \frac{\sigma a e \sin I}{n_{☾}^*-n_{\oplus}^*} \left\{ -\sin (h+g+\lambda_{\oplus}-\lambda_{\odot}) + \sin (h-g+\lambda_{\oplus}-\lambda_{\odot}) \right\}$$

$$\frac{\partial S_3' \text{ (radiation)}}{\partial e} = \frac{3}{2} \frac{\sigma a}{n_{☾}^*-n_{\oplus}^*} \left\{ \cos^2 \frac{I}{2} \sin (h+g+\lambda_{\oplus}-\lambda_{\odot}) + \sin^2 \frac{I}{2} \sin (h-g+\lambda_{\oplus}-\lambda_{\odot}) \right\}$$

$$\frac{\partial S_3' \text{ (radiation)}}{\partial g} = \frac{3}{2} \frac{\sigma a e}{n_{☾}^*-n_{\oplus}^*} \left\{ \cos^2 \frac{I}{2} \cos (h+g+\lambda_{\oplus}-\lambda_{\odot}) - \sin^2 \frac{I}{2} \cos (h-g+\lambda_{\oplus}-\lambda_{\odot}) \right\}$$

$$\frac{\partial S_3' \text{ (radiation)}}{\partial h} = \frac{3}{2} \frac{\sigma a e}{n_{☾}^*-n_{\oplus}^*} \left\{ \cos^2 \frac{I}{2} \cos (h+g+\lambda_{\oplus}-\lambda_{\odot}) + \sin^2 \frac{I}{2} \cos (h-g+\lambda_{\oplus}-\lambda_{\odot}) \right\}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial a} = \frac{3}{32}\,\frac{n_\oplus^2 a}{n_☾^* - n_\oplus^*}\{2(2+3e^2)\sin^2 I \sin 2(h+\lambda_\oplus - \lambda_\odot)$$

$$+ 5e^2(1+\cos I)^2 \sin 2(h+g+\lambda_\oplus - \lambda_\odot)$$

$$+ 5e^2(1-\cos I)^2 \sin 2(h-g+\lambda_\oplus - \lambda_\odot)\}$$

$$\frac{\partial S_3'(\text{sun})}{\partial I} = \frac{3}{32}\,\frac{n_\oplus^2 a^2 \sin I}{n_☾^* - n_\oplus^*}\{2(2+3e^2)\cos I \sin 2(h+\lambda_\oplus - \lambda_\odot)$$

$$- 5e^2 (1+\cos I) \sin 2(h+g+\lambda_\oplus - \lambda_\odot)$$

$$+ 5e^2 (1-\cos I) \sin 2(h-g+\lambda_\oplus - \lambda_\odot)\}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial e} = \frac{3}{32}\,\frac{n_\oplus^2 a^2 e}{n_☾^* - n_\oplus^*}\{6 \sin^2 I \sin 2(h+\lambda_\oplus - \lambda_\odot)$$

$$+ 5(1+\cos I)^2 \sin 2(h+g+\lambda_\oplus - \lambda_\odot)$$

$$+ 5(1-\cos I)^2 \sin 2(h-g+\lambda_\oplus - \lambda_\odot)\}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial g} = \frac{15}{32}\,\frac{n_\oplus^2 a^2 e^2}{n_☾^* - n_\oplus^*}\{(1+\cos I)^2 \cos 2(h+g+\lambda_\oplus - \lambda_\odot)$$

$$- (1-\cos I)^2 \cos 2(h-g+\lambda_\oplus - \lambda_\odot)\}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial h} = \frac{3}{32}\,\frac{n_\oplus^2 a^2}{n_☾^* - n_\oplus^*}\{2(2+3e^2)\sin^2 I \cos 2(h+\lambda_\oplus - \lambda_\odot)$$

$$+ 5e^2 (1+\cos I)^2 \cos 2(h+g+\lambda_\oplus - \lambda_\odot)$$

$$+ 5e^2 (1-\cos I)^2 \cos 2(h-g+\lambda_\oplus - \lambda_\odot)\}$$

$$\frac{\partial S_3'(e_☾)}{\partial a} = \frac{3}{32\epsilon} n_☾ e_☾ a \{-2(2+3e^2)(1-3\cos^2 I) + 30 e^2 \sin^2 I \cos 2g$$

$$+6(2+3e^2)\sin^2 I \cos 2h + 15 e^2 (1+\cos I)^2 \cos(2g+2h)$$

$$+15 e^2 (1-\cos I)^2 \cos(2g-2h)\} \sin l_\oplus$$

$$\frac{\partial S_3'(e_☾)}{\partial I} = \frac{9}{32\epsilon} n_☾ e_☾ a^2 \sin I \{-2(2+3e^2)\cos I + 10 e^2 \cos I \cos 2g$$

$$+2(2+3e^2)\cos I \cos 2h - 5e^2(1+\cos I)\cos(2g+2h)$$

$$+5e^2(1-\cos I)\cos(2g-2h)\} \sin l_\oplus$$

$$\frac{\partial S_3'(e_☾)}{\partial e} = \frac{9}{32\epsilon} n_☾ e_☾ a^2 e \{-2(1-3\cos^2 I) + 10 \sin^2 I \cos 2g$$

$$+6\sin^2 I \cos 2h + 5(1+\cos I)^2 \cos(2g+2h)$$

$$+5(1-\cos I)^2 \cos(2g-2h)\} \sin l_\oplus$$

$$\frac{\partial S_3'(e_☾)}{\partial g} = -\frac{45}{32\epsilon} n_☾ e_☾ a^2 e^2 \{2\sin^2 I \sin 2g + (1+\cos I)^2 \sin(2g+2h)$$

$$+(1-\cos I)^2 \sin(2g-2h)\} \sin l_\oplus$$

$$\frac{\partial S_3'(e_☾)}{\partial h} = -\frac{9}{32\epsilon} n_☾ e_☾ a^2 \{2(2+3e^2)\sin^2 I \sin 2h + 5e^2(1+\cos I)^2 \sin(2g+2h)$$

$$-5e^2(1-\cos I)^2 \sin(2g-2h)\} \sin l_\oplus$$

$$\frac{\partial S_3'(i_☾)}{\partial a} = -\frac{3}{4\epsilon}\,\frac{n_☾^2 a \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\,\{-2(2+3e^2)\cos I \sin h$$

$$+ 5e^2(1+\cos I)\sin(2g+h)$$

$$+ 5e^2(1-\cos I)\sin(2g-h)\}\cos v_\oplus$$

$$\frac{\partial S_3'(i_☾)}{\partial I} = -\frac{3}{8\epsilon}\,\frac{n_☾^2 a^2 \sin i_☾}{n_☾ + N_{\omega ☾}}\,\{2(2+3e^2)(1-2\cos^2 I)\sin h$$

$$-5e^2(1+\cos I)(1-2\cos I)\sin(2g+h)$$

$$+5e^2(1-\cos I)(1+2\cos I)\sin(2g-h)\}\cos v_\oplus$$

$$\frac{\partial S_3'(i_☾)}{\partial e} = -\frac{3}{4\epsilon}\,\frac{n_☾^2 a^2 e \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\,\{-6\cos I \sin h$$

$$+ 5(1+\cos I)\sin(2g+h)$$

$$+ 5(1-\cos I)\sin(2g-h)\}\cos v_\oplus$$

$$\frac{\partial S_3'(i_☾)}{\partial g} = -\frac{15}{4\epsilon}\,\frac{n_☾^2 a^2 e^2 \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\,\{(1+\cos I)\cos(2g+h) + (1-\cos I)\cos(2g-h)\}\cos v_\oplus$$

$$\frac{\partial S_3'(i_☾)}{\partial h} = -\frac{3}{8\epsilon}\,\frac{n_☾^2 a^2 \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\,\{-2(2+3e^2)\cos I \cos h$$

$$+ 5e^2(1+\cos I)\cos(2g+h)$$

$$- 5e^2(1-\cos I)\cos(2g-h)\}\cos v_\oplus$$

$$\frac{\partial S_3'(\oplus)}{\partial a} = 0$$

$$\frac{\partial S_3'(\oplus)}{\partial I} = -\frac{3}{16\epsilon}\frac{n_{\mathbb{C}}^2 j_2 R_\oplus^2 \sin I_\oplus}{n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}}}\{-2[\sin I\cos I\sin I_\oplus \sin 2(h+\lambda_\oplus - \bar{\Omega}')$$

$$-4(1-2\cos^2 I)\cos I_\oplus \sin(h+\lambda_\oplus - \bar{\Omega}')]$$

$$+\beta^2(1+2\sqrt{1-e^2})[\sin I(1+\cos I)\sin I_\oplus \sin 2(h+g+\lambda_\oplus - \bar{\Omega}')$$

$$-\sin I(1-\cos I)\sin I_\oplus \sin 2(h-g+\lambda_\oplus - \bar{\Omega}')$$

$$-4(1+\cos I)(1-2\cos I)\cos I_\oplus \sin(h+2g+\lambda_\oplus - \bar{\Omega}')$$

$$-4(1-\cos I)(1+2\cos I)\cos I_\oplus \sin(h-2g+\lambda_\oplus - \bar{\Omega}')]\}$$

$$\frac{\partial S_3'(\oplus)}{\partial e} = -\frac{3}{16\epsilon}\frac{n_{\mathbb{C}}^2 j_2 R_\oplus^2 \beta^2 \sin I_\oplus(2+\sqrt{1-e^2})}{e(n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}})}\{-(1+\cos I)^2\sin I_\oplus \sin 2(h+g+\lambda_\oplus - \bar{\Omega}')$$

$$-(1-\cos I)^2 \sin I_\oplus \sin 2(h-g+\lambda_\oplus - \bar{\Omega}')$$

$$+8\sin I(1+\cos I)\cos I_\oplus \sin(h+2g+\lambda_\oplus - \bar{\Omega}')$$

$$-8\sin I(1-\cos I)\cos I_\oplus \sin(h-2g+\lambda_\oplus - \bar{\Omega}')\}$$

$$\frac{\partial S_3'(\oplus)}{\partial g} = -\frac{3}{16\epsilon}\frac{n_{\mathbb{C}}^2 j_2 R_\oplus^2 \beta^2 \sin I_\oplus(1+2\sqrt{1-e^2})}{e(n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}})}\{-(1+\cos I)^2\sin I_\oplus \cos 2(h+g+\lambda_\oplus - \bar{\Omega}')$$

$$+(1-\cos I)^2 \sin I_\oplus \cos 2(h-g+\lambda_\oplus - \bar{\Omega}')$$

$$+8\sin I(1+\cos I)\cos I_\oplus \cos(h+2g+\lambda_\oplus - \bar{\Omega}')$$

$$+8\sin I(1-\cos I)\cos I_\oplus \cos(h-2g+\lambda_\oplus - \bar{\Omega}')\}$$

$$\frac{\partial S_3'(\oplus)}{\partial h} = -\frac{3}{16\epsilon}\frac{n_{\mathbb{C}}^2 j_2 R_\oplus^2 \sin I_\oplus}{n_{\mathbb{C}}^* - N_{\Omega_{\mathbb{C}}}}\{-2\sin I[\sin I \sin I_\oplus \cos 2(h+\lambda_\oplus - \overline{\Omega}')$$

$$+4\cos I \cos I_\oplus \cos(h+\lambda_\oplus - \overline{\Omega}')]$$

$$+\beta^2(1+2\sqrt{1-e^2})[-(1+\cos I)^2 \sin I_\oplus \cos 2(h+g+\lambda_\oplus - \overline{\Omega}')$$

$$-(1-\cos I)^2 \sin I_\oplus \cos 2(h-g+\lambda_\oplus - \overline{\Omega}')$$

$$+4\sin I(1+\cos I)\cos I_\oplus \cos(h+2g+\lambda_\oplus - \overline{\Omega}')$$

$$-4\sin I(1-\cos I)\cos I_\oplus \cos(h-2g+\lambda_\oplus - \overline{\Omega}')]\}$$

$$\frac{\partial S_3'(\text{lib.})}{\partial a} = 0$$

$$\frac{\partial S_3'(\text{lib.})}{\partial I} = -2\alpha G \sin I \sin \ell_\odot$$

$$\frac{\partial S_3'(\text{lib.})}{\partial e} = 0$$

$$\frac{\partial S_3'(\text{lib.})}{\partial g} = 0$$

$$\frac{\partial S_3'(\text{lib.})}{\partial h} = 0$$

26. Secular Perturbations and Perturbations Depending Strictly on g''

At this stage of the problem the Hamiltonian is

$$\mathcal{F}'' = \mathcal{F}_0'' + \mathcal{F}_1'' + \mathcal{F}_2'' + \mathcal{F}_3''$$

where

$$\mathcal{F}_0'' = \frac{\mu^2}{2L'^2} - T = \text{const.}$$

$$\mathcal{F}_1'' = n_{\mathbb{C}}^* H'' = \text{const.}$$

$$\mathcal{F}_2'' = F_2''(-, g'', -, L', G'', H'')$$

$$\mathcal{F}_3'' = F_3''(-, g'', -, L', G'', H'')$$

and L', T and H'' are constants in time. T is effectively a 3rd order quantity but in the manipulation of von Zeipel's method its contributions are of the zero order, for it is a variable itself, independent of all the others.

It is easy to see that the variable part of the Hamiltonian $\mathcal{F}''$, which will generate the equations of motion, is factored by small parameters. Therefore, a method of successive approximations such as von Zeipel's cannot be applied.

Therefore, the system of differential equations produced by $\mathcal{F}''$ must be integrated directly or by using some kind of approximation. The 3rd order part, $\mathcal{F}_3''$, will be neglected for the moment. The remaining Hamiltonian is

$$F'' = \mathcal{F}_0'' + F_1'' + F_2'' = \text{const.}$$

where, since L', H'' and T are constants, $F_2'' = \text{const.}$ The equations of motion are

$$\dot{G}'' = \frac{\partial F_2''}{\partial g''} = \dot{G}''(g'', G'')$$

$$\dot{g}'' = -\frac{\partial F_2''}{\partial G''} = \dot{g}''(g'', G'')$$

$$\dot{h}'' = -n_{\mathbb{C}}^* - \frac{\partial F_2''}{\partial H''} = \dot{h}''(g'', G'')$$

$$\dot{l}'' = n' - \frac{\partial F_2''}{\partial L'} = \dot{l}''(g'', G'') .$$

The first two equations represent a system of one degree of freedom since F_2'' is a first integral. Then it can be integrated and therefore h'' and l'' are obtained by quadrature.

The approach taken in this section is similar to the one used by Kozai in Reference 6 . The equations for $\dot{\eta}''$ and $\dot{g}''$ are found to be

$$\dot{\eta}'' = -5 K_1 (1 - \eta''^2)(1 - \theta''^2) \sin 2g''$$

$$\dot{g}'' = -\frac{K_1}{\eta''} [(\eta''^2 - 5\theta''^2) - 5(\eta''^2 - \theta''^2) \cos 2g''] - \frac{K_2}{L' \eta''^4} (1 - 5\theta''^2)$$

and the integral $6F_2''$ is written

$$K_1 L' [(5 - 3\eta''^2)(-1 + 3\theta''^2) + 15(1 - \eta''^2)(1 - \theta''^2) \cos 2g'']$$

$$-\frac{2K_2}{\eta''^3}(1 - 3\theta''^2) = C = \text{const.} = 6F_2'',$$

where

$$K_1 = \frac{3}{8\epsilon} n' \left(\frac{n_{☾}}{n'}\right)^2 = \frac{3}{4\epsilon} n' \alpha_1 , \qquad \alpha_1 = \frac{1}{2}\left(\frac{a'}{a_{☾}}\right)^3$$

$$K_2 = \frac{3}{4} n'^2 b^2 J_2 = \frac{3}{4} n' L \alpha_2 , \qquad \alpha_2 = \frac{b^2 J_2}{a'^2}$$

$$\eta'' = \frac{G''}{L'}$$

$$\theta'' = \frac{H''}{G''} .$$

In order to integrate the system, the following remarks are made:

(1) η'' is assumed to be a periodic function of time of the form

$$\eta'' = \eta_0'' + \Delta\eta''(\alpha_1, \alpha_2; t),$$

where η_0'' is a constant and $\Delta\eta''$ strictly periodic in t and such that $\Delta\eta''(0, 0; t) = 0$ for any $\underline{t}$, so that

$$\Delta\eta'' = \alpha_1 \Delta\eta_1''(\alpha_1, \alpha_2; t) + \alpha_2 \Delta\eta_2''(\alpha_1, \alpha_2; t).$$

(2) g'' is assumed to be the sum of a linear function of time plus periodic terms, i.e., of the form

$$g'' = g_0'' + n_g(\alpha_1, \alpha_2)t + \Delta g''(\alpha_1, \alpha_2; t),$$

where g_0'' is a constant, n_g is a constant which can eventually be zero (libration cases) and $\Delta g''$ is strictly periodic in t. They are such that $n_g(0, 0) = 0$ and $\Delta g''(0, 0; t) = 0$ for any t. Thus

$$n_g = \alpha_1 n_1(\alpha_1, \alpha_2) + \alpha_2 n_2(\alpha_1, \alpha_2)$$

$$\Delta g'' = \alpha_1 \Delta g_1''(\alpha_1, \alpha_2; t) + \alpha_2 \Delta g_2''(\alpha_1, \alpha_2; t).$$

(3) If assumption (1) is correct, we have

$$\dot{\eta}'' = \alpha_1 \Delta\dot{\eta}_1'' + \alpha_2 \Delta\dot{\eta}_2'',$$

and, comparing with the equation for $\dot{\eta}''$, it is seen that $\Delta\dot{\eta}_2''$ must be zero, or say $\Delta\eta''(0, \alpha_2; t) = 0$ for any α_2 and t. Then η'' will depend on α_2 through a fourth order term of the form $\alpha_1\alpha_2 \cdots$. One concludes that if α_2 is neglected in the evaluation of η'', the error will possibly be a periodic function factored by the product $\alpha_1\ \alpha_2$, and it is tolerable.

The same is not true for g'' since neglecting terms $\alpha_1\alpha_2 \cdots$ may not be tolerable due to the presence of a linear term in time.

27. The Integration of $\dot{\eta}''$

Rewriting the integral F_2'' we have

$$(5 - 3\eta''^2)(-1 + 3\theta''^2) + 15(1 - \eta''^2)(1 - \theta''^2)\cos 2g'' =$$

$$= \frac{1}{K_1 L'}\,[C + 2K_2\,\eta''^{-3}(1 - 3\theta''^2)] = W.$$

If η_1'' is the value of η'' for $g'' = 0$, then neglecting the oblateness terms we have

$$W = 10 - 12\eta_1''^2 + 6\nu^2 = \frac{C}{K_1 L'} = \text{const.},$$

where

$$\nu = \frac{H''}{L'} = \text{const.}$$

We are going to evaluate $\cos 2g''$ from this integral in order to find the value of dx/dt. The error in doing this is of the fourth order by remark (3) in the previous section. Therefore,

$$\cos 2g'' = \frac{2(5 - 6x_1 + 3\nu^2) + (5 - 3x)(1 - 3\nu^2 x^{-1})}{15(1 - x)(1 - \nu^2 x^{-1})},$$

where $x = \eta''^2$ and $x_1 = \eta_1''^2$. Note that the definition of $\eta_1''^2$ does not necessarily correspond to a physical situation.

The substitution of $\cos 2g''$ into the equations for $\dot{\eta}''$ gives

$$\frac{dx}{dt} = \mp\, 4\sqrt{6}\,K_1\,[(x - x_1)(x - x_2)(x - x_3)]^{1/2}.$$

Also

$$\Delta_1 G'' = \int_0^t \frac{\partial F_2''}{\partial g} \, dt = L_0'' (\sqrt{x} - \sqrt{x_0}) .$$

In the equation for dx/dt the minus sign is used when sin $2g''$ is positive and the plus sign is used when sin $2g''$ is negative. The choice of sign will be treated in the next section.

The values of x_2 and x_3 are defined by the relations

$$x_2 \, x_3 = \frac{5}{3} \nu^2$$

$$x_2 + x_3 = \frac{1}{3} (5 + 5\nu^2 - 2x_1)$$

and, for eccentricity $e'' < 1$, they are always real.

The solution for x will be generally expressed by elliptic functions, except in a few cases where x_1 , x_2 , x_3 assume special values.

The possible cases for the relative values of x_1 , x_2 , x_3 are next determined based on the fact that one looks for a real solution of the problem. Any complex solution, from the physical point of view, must be discarded.

It is noted that, since

$$x = \eta''^2 = 1 - e''^2$$

$$\nu^2 = \frac{H''^2}{L'^2} = (1 - e''^2) \cos^2 I'',$$

x and ν satisfy the inequalities

$$1 \geq x \geq \nu^2 \geq 0 .$$

Furthermore, if the solution x has to be real, then from the differential equation for x there is the additional condition

$$(x - x_1)(x - x_2)(x - x_3) \geq 0.$$

(a) $\underline{x_1 = 1}$

Then

$$x_2 = 1, \; x_3 = \frac{5}{3}\nu^2.$$

If

$$\nu^2 = \frac{3}{5}, \; x_3 = 1,$$

then

$$x_1 = x_2 = x_3 = 1,$$

and since $x \leq 1$, the only possibility is

$$x = 1 = \text{const.}$$

or

$$e'' = \text{const.} = 0$$

$$\cos I'' = \pm\sqrt{3/5} = \text{const.}$$

If

$$\nu^2 > \frac{3}{5}, \; x_3 > 1,$$

then

$$1 = x_1 = x_2 < x_3,$$

and again the only possibility is

$$x = 1 = \text{const.}$$

or

$$e'' = \text{const.} = 0$$

$$|\cos I''| \geq \sqrt{3/5}.$$

If

$$\nu^2 < \frac{3}{5}, \; x_3 < 1,$$

then

$$1 = x_1 = x_2 > x_3$$

and

$$x_3 \leq x \leq 1.$$

Thus,

$$\frac{5}{3} \nu^2 \leq x \leq 1$$

or

$$0 \leq e'' \leq 1$$

$$0 \leq \cos I'' \leq 1.$$

(b) $\underline{x_1 = \nu^2}$

Then

$$x_3 = \nu^2 = x_1$$

$$x_2 = \frac{5}{3} > 1$$

and

$$0 \leq x_3 = x_1 = \nu^2 \leq 1.$$

The only possibility is

$$x = \nu^2 = \text{const.}$$

or

$$\cos I'' = 1 \text{ and/or } e'' = 1.$$

(c) $\underline{\nu^2 < x_1 \neq 1}$

Thus,

$$\nu^2 < x_2 < \frac{5}{3}, \quad x_2 \neq 1$$

and

$$\nu^2 < x_3 < \frac{5}{3}, \quad x_3 \neq \frac{5}{3} \nu^2.$$

This is the most general case.

28. General Case ($\nu^2 < x_1 \neq 1$)

In this case, x_1, x_2, x_3 will be ordered in ascending order and will be called x_1', x_2', x_3' so that $x_3' > x_2' > x_1'$.

The solution for x is expressed in terms of Jacobi's elliptic functions as

$$x = x_1' + (x_2' - x_1')\left[\frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \mp \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}\right]^2,$$

where

$$k = +\sqrt{\frac{x_2' - x_1'}{x_3' - x_1'}}$$

$$v = 2\sqrt{6}\, K_1 \sqrt{x_3' - x_1'}\ \ t$$

$$\operatorname{sn} u = +\sqrt{\frac{x_0 - x_1'}{x_2' - x_1'}}$$

$$\operatorname{cn} u = +\sqrt{\frac{x_2' - x_0}{x_2' - x_1'}}$$

$$\operatorname{dn} u = +\sqrt{\frac{x_3' - x_0}{x_3' - x_1'}}$$

and $\mathrm{cn}\, v$, $\mathrm{dn}\, v$, $\mathrm{sn}\, v$ are the Jacobi's elliptic functions with modulus $\underline{k}$ and argument $\underline{v}$; x_0 is the value of x at the intial time which is taken equal to zero.

Since at this stage g'' is unknown, the choice of the sign can be made only after the equation for $\dot{g}''$ is integrated. That is, the correct sign is determined by a comparison between the value of the integral F_2'' using $\cos 2g''$ obtained from the integration and its value at $t = t_0$. The solution for x given above is one in which the oblateness portion of the energy integral has been neglected. By an iteration process, the correction due to oblateness can be taken into account.

29. The Integration of $\dot{g}''$

Once x is a known function of time, $\dot{g}''$ can be integrated using $\underline{x}$ as the independent variable. In fact, consider the transformation

$$\frac{dg''}{dt} = \frac{dg''}{dx}\frac{dx}{dt} = \mp 4\sqrt{6}\, K_1 \sqrt{Q(x)}\, \frac{dg''}{dx},$$

where

$$Q(x) = (x - x_1)(x - x_2)(x - x_3).$$

On the other hand,

$$\dot{g}'' = -\frac{K_1}{\sqrt{x}}\left[(x - 5\nu^2 x^{-1}) - 5(x - \nu^2 x^{-1}) \cos 2g''\right] - \frac{K_2}{L' x^2}(1 - 5\nu^2 x^{-1})$$

and

$$\cos 2g'' = \frac{1}{15\,(1-x)\,(1-\nu^2 x^{-1})}\left\{\frac{C}{K_1 L'} + 2\,\frac{K_2}{K_1 L'}\,\frac{1-3\nu^2 x^{-1}}{x^{3/2}} + (5-3x)\,(1-3\nu^2 x^{-1})\right\},$$

so that

$$\dot{g}'' = -K_1\sqrt{x} + 5\,K_1\,\nu^2 x^{-3/2} + \frac{C}{3L'}\,\frac{x-\nu^2 x^{-1}}{\sqrt{x}\,(1-x)\,(1-\nu^2 x^{-1})}$$

$$+\frac{2K_2}{3L'}\,\frac{(x-\nu^2 x^{-1})(1-3\nu^2 x^{-1})}{x^2\,(1-x)\,(1-\nu^2 x^{-1})}$$

$$+\frac{K_1}{3}\,\frac{(x-\nu^2 x^{-1})\,(5-3x)\,(1-3\nu^2 x^{-1})}{\sqrt{x}\,(1-x)\,(1-\nu^2 x^{-1})}$$

$$-\frac{K_2}{L'}\,\frac{1-5\nu^2 x^{-1}}{x^2}\,.$$

After reduction, this becomes

$$\dot{g}'' = \frac{C}{3L'}\left\{\frac{1}{(1-x)\sqrt{x}} - \frac{\nu^2}{(x-\nu^2)\sqrt{x}} - \frac{1}{\sqrt{x}}\right\} + \frac{2}{3}\,\frac{K_2}{L'}\left\{(1-3\nu^2)\frac{1}{1-x}\right.$$

$$\left. + \frac{2}{\nu^2}\,\frac{1}{x-\nu^2} + \frac{\nu^2-3\nu^4-2}{\nu^2}\,\frac{1}{x} - \frac{7+6\nu^2}{2}\,\frac{1}{x^2} + \frac{9}{2}\,\frac{\nu^2}{x^3}\right\}$$

$$+\frac{K_1}{3}\left\{2(1-3\nu^2)\frac{1}{(1-x)\sqrt{x}}-2\nu^2(3\nu^2-5)\frac{1}{(x-\nu^2)\sqrt{x}}\right.$$

$$\left.-2(3\nu^2+1)\frac{1}{\sqrt{x}}\right\}$$

or, finally,

$$\mp 4\sqrt{6}\,K_1\frac{dg''}{dx}=\frac{1}{3}\left[\frac{C}{L'}+2(1-3\nu^2)K_1\right]\frac{1}{(1-x)\sqrt{x\,Q(x)}}$$

$$-\frac{1}{3}\left[\frac{C\nu^2}{L'}+2\nu^2(3\nu^2-5)K_1\right]\frac{1}{(x-\nu^2)\sqrt{x\,Q(x)}}$$

$$-\frac{1}{3}\left[\frac{C}{L'}+2(1+3\nu^2)K_1\right]\frac{1}{\sqrt{x\,Q(x)}}$$

$$+\frac{2}{3}\frac{K_2}{L'}\left\{(1-3\nu^2)\frac{1}{(1-x)\sqrt{Q(x)}}+\frac{2}{\nu^2}\frac{1}{(x-\nu^2)\sqrt{Q(x)}}\right.$$

$$+\frac{\nu^2-3\nu^4-2}{\nu^2}\frac{1}{x\sqrt{Q(x)}}-\frac{7+6\nu^2}{2}\frac{1}{x^2\sqrt{Q(x)}}$$

$$\left.+\frac{9\nu^2}{2}\frac{1}{x^3\sqrt{Q(x)}}\right\}.$$

The integration gives

$$\mp 4\sqrt{6}K_1 \Delta_1 g'' = \frac{1}{3}\left[\frac{C}{L'} + 2(1-3\nu^2)K_1\right]\int_{x_0}^{x} \frac{dx}{(1-x)\sqrt{x\,Q(x)}}$$

$$-\frac{1}{3}\left[\frac{C\nu^2}{L'} + 2\nu^2(3\nu^2-5)K_1\right]\int_{x_0}^{x} \frac{dx}{(x-\nu^2)\sqrt{x\,Q(x)}}$$

$$-\frac{1}{3}\left[\frac{C}{L'} + 2(1+3\nu^2)K_1\right]\int_{x_0}^{x} \frac{dx}{\sqrt{x\,Q(x)}}$$

$$+\frac{2}{3}\frac{K_2}{L'}(1-3\nu^2)\int_{x_0}^{x} \frac{dx}{(1-x)\sqrt{Q(x)}} + \frac{4K_2}{3L'\,\nu^2}\int_{x_0}^{x} \frac{dx}{(x-\nu^2)\sqrt{Q(x)}}$$

$$+\frac{2}{3}\frac{K_2}{L'}\frac{\nu^2-3\nu^4-2}{\nu^2}\int_{x_0}^{x} \frac{dx}{x\sqrt{Q(x)}} - \frac{1}{3}\frac{K_2}{L'}(7+6\nu^2)\int_{x_0}^{x} \frac{dx}{x^2\sqrt{Q(x)}}$$

$$+\frac{3\nu^2 K_2}{L'}\int_{x_0}^{x} \frac{dx}{x^3\sqrt{Q(x)}},$$

where

$$\int_{x_0}^{x} f(x)dx = \int_{x_1}^{x} f(x)dx - \int_{x_1}^{x_0} f(x)dx.$$

The various elliptic integrals needed are given in Appendix A.

30. The Variables l'' and h''

Once x is known, l'' and h'' are integrated in the same manner as was done for g''. The original equations are

$$\dot{l}'' = -\frac{\partial F''}{\partial L'} = n' - \frac{n'}{8\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \{ (10-3x)(-1+3\nu^2 x^{-1})$$

$$+ 15(2-x)(1-\nu^2 x^{-1}) \cos 2g'' \} - \frac{3}{4}\frac{n'^2}{L'} b^2 J_2 (1-3\nu^2 x^{-1}) x^{-3/2}$$

$$\dot{h}'' = -\frac{\partial F''}{\partial H''} = -n^*_{\mathbb{C}} - \frac{3}{8\epsilon}\frac{n'\nu}{x}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 (5-3x)$$

$$-\frac{3}{2} n'^2 b^2 J_2 \frac{\nu}{L'} x^{-5/2} + \frac{15}{8\epsilon}\frac{n'\nu}{x}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 (1-x)\cos 2g'',$$

where $\cos 2g''$ is to be replaced by the formula in the previous section. Therefore

$$\dot{l}'' = n' - \frac{1}{3} K_1 (10-3x)(-1+3\nu^2 x^{-1}) - \frac{C}{3L'}\frac{2-x}{1-x}$$

$$-\frac{2}{3}\frac{K_2}{L'}\frac{(2-x)(1-3\nu^2 x^{-1})}{x^{3/2}(1-x)} - \frac{K_1}{3}\frac{(2-x)(5-3x)(1-3\nu^2 x^{-1})}{1-x}$$

$$-\frac{K_2}{L'}\frac{1-3\nu^2 x^{-1}}{x^{3/2}}$$

$$\dot{h}'' = -n_{\mathfrak{C}}^{*} - K_1 \nu \frac{5 - 3x}{x} - 2K_2 \frac{\nu}{L'} \frac{1}{x^{5/2}}$$

$$+ \frac{C\nu}{3L'} \frac{1}{x(1 - \nu^2 x^{-1})} + \frac{2}{3} \frac{K_2 \nu}{L'} \frac{1 - 3\nu^2 x^{-1}}{x^{5/2}(1 - \nu^2 x^{-1})}$$

$$+ \frac{K_1 \nu}{3} \frac{(5 - 3x)(1 - 3\nu^2 x^{-1})}{x(1 - \nu^2 x^{-1})} .$$

After reduction, these become

$$\dot{l}'' = \left[n' + \frac{2}{3} K_1 - \frac{C}{3L'}\right] - \frac{C}{3L'} \frac{1}{1 - x} - \frac{K_1}{3} \frac{2 - 6\nu^2}{1 - x}$$

$$- \frac{2}{3} \frac{K_2}{L'} \left[\frac{7 - 6\nu^2}{2} \frac{1}{x\sqrt{x}} + \frac{1 - 3\nu^2}{(1 - x)\sqrt{x}} - \frac{21\nu^2}{2} \frac{1}{x^2\sqrt{x}}\right]$$

$$\dot{h}'' = [-n_{\mathfrak{C}}^{*} + 2\nu K_1] + \left[\frac{2}{3} K_1 \nu (3\nu^2 - 5) + \frac{C\nu}{3L'}\right] \frac{1}{x - \nu^2}$$

$$+ \frac{2}{3} \frac{K_2 \nu}{L'} \left[\frac{2}{\nu^2} \frac{1}{x\sqrt{x}} - \frac{2}{\nu^2} \frac{1}{(x - \nu^2)\sqrt{x}}\right].$$

Finally, we have

$$\mp 4\sqrt{6}\, K_1 \frac{dl''}{dx} = \left[n' + \frac{2}{3} K_1 - \frac{C}{3L'}\right] \frac{1}{\sqrt{Q(x)}} - \left[\frac{C}{3L'}\right.$$

$$+\frac{K_1}{3}(2-6\nu^2)\Big]\frac{1}{(1-x)\sqrt{Q(x)}}-\frac{2}{3}\frac{K_2}{L'}\left[\frac{7-6\nu^2}{2}\frac{1}{x\sqrt{xQ(x)}}\right.$$

$$\left.+\frac{1-3\nu^2}{(1-x)\sqrt{xQ(x)}}-\frac{21\nu^2}{2}\frac{1}{x^2\sqrt{xQ(x)}}\right]$$

and

$$\mp 4\sqrt{6}\,K_1\frac{dh''}{dx}=(-n_{\mathbb{C}}^*+2\nu K_1)\frac{1}{\sqrt{Q(x)}}+\left[\frac{2}{3}K_1\nu(3\nu^2-5)+\frac{C\nu}{3L'}\right]\frac{1}{(x-\nu^2)\sqrt{Q(x)}}$$

$$+\frac{2}{3}\frac{K_2\nu}{L'}\left[\frac{2}{\nu^2}\frac{1}{x\sqrt{xQ(x)}}-\frac{2}{\nu^2}\frac{1}{(x-\nu^2)\sqrt{xQ(x)}}\right]$$

which, integrated, give

$$\mp 4\sqrt{6}K_1\Delta_1\ell''=\mp 4\sqrt{6}K_1\left(n'+\frac{2}{3}K_1-\frac{C}{3L'}\right)t-\left[\frac{K_1(2-6\nu^2)}{3}+\frac{C}{3L'}\right]\int_{x_0}^{x}\frac{dx}{(1-x)\sqrt{Q(x)}}$$

$$-\frac{K_2}{3L'}(7-6\nu^2)\int_{x_0}^{x}\frac{dx}{x\sqrt{xQ(x)}}+7\nu^2\frac{K_2}{L'}\int_{x_0}^{x}\frac{dx}{x^2\sqrt{xQ(x)}}$$

$$-\frac{2}{3}\frac{K_2}{L'}(1-3\nu^2)\int_{x_0}^{x}\frac{dx}{(1-x)\sqrt{xQ(x)}}$$

and

$$\mp 4\sqrt{6}K_1\Delta_1 h''=\mp 4\sqrt{6}\,K_1(-n_{\mathbb{C}}^*+2K_1\nu)t+\left[\frac{2K_1\nu}{3}(3\nu^2-5)+\frac{C\nu}{3L'}\right]\int_{x_0}^{x}\frac{dx}{(x-\nu^2)\sqrt{Q(x)}}$$

$$-\frac{4K_2}{3\nu L'}\left[\int_{x_0}^{x}\frac{dx}{(x-\nu^2)\sqrt{xQ(x)}}-\int_{x_0}^{x}\frac{dx}{x\sqrt{xQ(x)}}\right].$$

The various elliptic integrals needed are given in Appendix A.

31. Special Cases

The following special case remains to be treated:

$x_1 = 1,\ \nu^2 < \frac{3}{5}$

Since $x_1 = x_2 > x_3$, the equation for x becomes

$$\frac{dx}{dt} = \mp 4\sqrt{6}\, K_1\, (1 - x)\sqrt{x - x_3}$$

whose solution is

$$x = x_3 + (1 - x_3)\left\{\frac{1 + \Xi \exp\left[\mp 4\sqrt{6}\, K_1\, (-\sqrt{1 - x_3}\ t)\right]}{1 - \Xi \exp\left[\mp 4\sqrt{6}\, K_1\, (-\sqrt{1 - x_3}\ t)\right]}\right\}^2$$

where

$$\Xi = \frac{\sqrt{x_0 - x_3} - \sqrt{1 - x_3}}{\sqrt{x_0 - x_3} + \sqrt{1 - x_3}}$$

and the choice of the sign $-$ or $+$ is done as in the general case after integration of the equation for $\dot{g}''$.

The equations for $\dot{g}''$, $\dot{l}''$ and h'' are the same as for the general case with the exception that in $Q(x)$, $x_1 = x_2 = 1$, so that the integration is essentially different. In fact, $\sqrt{Q(x)} = (1 - x)\sqrt{x - x_3}$, so that

$$\mp 4\sqrt{6}\, K_1 \frac{dg''}{dx} = +\frac{1}{3}\left[\frac{C}{L'} + 2(1 - 3\nu^2)\, K_1\right] \frac{1}{(1 - x)^2 \sqrt{x(x - x_3)}}$$

$$- \frac{1}{3}\left[\frac{C\nu^2}{L'} + 2\nu^2\,(3\nu^2 - 5)\, K_1\right] \frac{1}{(x - \nu^2)\,(1 - x)\sqrt{x(x - x_3)}}$$

$$-\frac{1}{3}\left[\frac{C}{L'} + 2\,(1 + 3\nu^2)\, K_1\right] \frac{1}{(1 - x)\sqrt{x(x - x_3)}}$$

$$+\frac{2}{3}\frac{K_2}{L'}\left\{+ (1 - 3\nu^2)\frac{1}{(1 - x)^2\sqrt{x - x_3}} + \frac{2}{\nu^2}\frac{1}{(x - \nu^2)\,(1 - x)\sqrt{x - x_3}}\right.$$

$$+\frac{\nu^2 - 3\nu^4 - 2}{\nu^2}\frac{1}{x(1 - x)\sqrt{x - x_3}} - \frac{7 + 6\nu^2}{2}\frac{1}{x^2(1 - x)\sqrt{x - x_3}}$$

$$\left. + \frac{9\nu^2}{2}\frac{1}{x^3\,(1 - x)\sqrt{x - x_3}}\right\}$$

$$\mp 4\sqrt{6}\, K_1 \frac{dl''}{dx} = \left[n' + \frac{2}{3}K_1 - \frac{C}{3L'}\right] \frac{1}{(1 - x)\sqrt{x - x_3}} - \left[\frac{C}{3L'} + \frac{K_1}{3}(2 - 6\nu^2)\right]$$

$$\cdot \frac{1}{(1 - x)^2\sqrt{x - x_3}} - \frac{2}{3}\frac{K_2}{L'}\left[\frac{7 - 6\nu^2}{2}\frac{1}{x(1 - x)\sqrt{x(x - x_3)}}\right.$$

$$\left. + (1 - 3\nu^2)\frac{1}{(1 - x)^2\sqrt{x(x - x_3)}} - \frac{21\nu^2}{2}\frac{1}{x^2(1 - x)\sqrt{x(x - x_3)}}\right]$$

$$\mp 4\sqrt{6}\,K_1 \frac{dh''}{dx} = (-n^*_{\mathbb{C}} + 2\nu K_1) \quad \frac{1}{(1-x)\sqrt{x-x_3}} + \left[\frac{2}{3} K_1 \nu (3\nu^2 - 5) + \frac{C\nu}{3L'}\right] \cdot$$

$$\cdot \frac{1}{(x-\nu^2)(1-x)\sqrt{x-x_3}} + \frac{2}{3}\frac{K_2 \nu}{L'} \left[\frac{2}{\nu^2} \frac{1}{x(1-x)\sqrt{x(x-x_3)}} \right.$$

$$\left. - \frac{2}{\nu^2} \frac{1}{(x-\nu^2)(1-x)\sqrt{x(x-x_3)}} \right] .$$

The result of the integration is

$$\mp 4\sqrt{6} K_1 \Delta_1 g'' = +\frac{1}{3}\left[\frac{C}{L'} + 2(1-3\nu^2) K_1\right] \int_{x_0}^{x} \frac{dx}{(1-x)^2 \sqrt{x(x-x_3)}}$$

$$-\frac{1}{3}\left[\frac{C\nu^2}{L'} + 2\nu^2 (3\nu^2 - 5) K_1\right] \int_{x_0}^{x} \frac{dx}{(x-\nu^2)(1-x)\sqrt{x(x-x_3)}}$$

$$-\frac{1}{3}\left[\frac{C}{L'} + 2(1+3\nu^2) K_1\right] \int_{x_0}^{x} \frac{dx}{(1-x)\sqrt{x(x-x_3)}}$$

$$+\frac{2}{3}\frac{K_2}{L'}(1-3\nu^2) \int_{x_0}^{x} \frac{dx}{(1-x)^2\sqrt{x-x_3}}$$

$$+\frac{4}{3}\frac{K_2}{L'\nu^2} \int_{x_0}^{x} \frac{dx}{(x-\nu^2)(1-x)\sqrt{x-x_3}} + \frac{2}{3}\frac{K_2}{L'\nu^2}(\nu^2 - 3\nu^4 - 2) \int_{x_0}^{x} \frac{dx}{x(1-x)\sqrt{x-x_3}}$$

$$-\frac{K_2}{L'} \cdot \frac{7+6\nu^2}{3} \int_{x_0}^{x} \frac{dx}{x^2(1-x)\sqrt{x-x_3}} + \frac{3K_2\nu^2}{L'} \int_{x_0}^{x} \frac{dx}{x^3(1-x)\sqrt{x-x_3}}$$

$$\mp 4\sqrt{6}K_1 \Delta_1 \ell'' = \mp 4\sqrt{6}\, K_1 \left(n' + \frac{2}{3}K_1 - \frac{C}{3L'}\right) t - \left[\frac{C}{3L'} + \frac{K_1}{3}(2 - 6\nu^2)\right] \int_{x_0}^{x} \frac{dx}{(1-x)^2 \sqrt{x - x_3}}$$

$$- \frac{K_2}{L'} \cdot \frac{7 - 6\nu^2}{3} \int_{x_0}^{x} \frac{dx}{x(1-x)\sqrt{x(x - x_3)}} - \frac{2}{3}\frac{K_2}{L'}(1 - 3\nu^2) \int_{x_0}^{x} \frac{dx}{(1-x)^2\sqrt{x(x-x_3)}}$$

$$+ 7\frac{K_2 \nu^2}{L'} \int_{x_0}^{x} \frac{dx}{x^2(1-x)\sqrt{x(x-x_3)}}$$

$$\mp 4\sqrt{6}K_1 \Delta_1 h'' = \mp 4\sqrt{6}\, K_1 (-n_{\mathfrak{C}}^{*} + 2\nu K_1) t$$

$$+ \left[\frac{2}{3}K_1 \nu (3\nu^2 - 5) + \frac{C\nu}{3L'}\right] \int_{x_0}^{x} \frac{dx}{(x - \nu^2)(1-x)\sqrt{x - x_3}}$$

$$+ \frac{4}{3}\frac{K_2}{L'\nu} \int_{x_0}^{x} \frac{dx}{x(1-x)\sqrt{x(x-x_3)}} - \frac{4}{3}\frac{K_2}{L'\nu} \int_{x_0}^{x} \frac{dx}{(x-\nu^2)(1-x)\sqrt{x(x-x_3)}} .$$

The various integrals needed are given in Appendix B.

This ends the particular cases to be considered. In resume, the precision of the results is as follows:

(1) In $\cos 2g''$ to integrate x: error is proportional to α_2

(2) In the integrated value of x: error is proportional to $\alpha_1\alpha_2$, if no iteration is done to improve x.

(3) In the integration of $\dot{g}''$, $\dot{h}''$, $\dot{l}''$: error is proportional to $\alpha_1^2\alpha_2$ or $\alpha_1\alpha_2^2$. This means that the motion of the pericenter and node and the perturbations in the mean anomaly are, at maximum, in error by a sixth order quantity or about 10^{-12}. If the time elapsed is not too large, these errors are tolerable.

32. Higher Order Perturbations

The solution of the problem up to this point stands as follows:

(1) Secular terms: correct to 2nd order, included in the integration of l'', g'', h''.

(2) Long period terms: correct to 2nd order in terms depending on h'' and t (and eventually g'').

(3) Long period terms: correct to 1st order in terms depending strictly on g'' and $\lambda_\odot$ and $\lambda_☾$.

In order to evaluate long period perturbations correct to 2nd order in terms depending on g'', the 3rd order part of the Hamiltonian should be included.

Consider then the equations

$$\Delta_2 \dot{l}'' = -\frac{\partial F_3''}{\partial L'} = f_1 \,(g'', G'')$$

$$\Delta_2 \dot{g}'' = -\frac{\partial F_3''}{\partial G''} = f_2 \,(g'', G''),$$

$$\Delta_2 \dot{h}'' = -\frac{\partial F_3''}{\partial H''} = f_3 \,(g'', G'') .$$

The functions f_i contain variables g'', G'' factored by 3rd order parameters ($\sim 10^{-6}$). The variables g'' and $G''(x)$ depend on time through very complicated functions. The substitution of the independent variable in terms of x is out of the question, since t is a function of x through the inverse Jacobian function sn^{-1}. The only possibility is then to consider $G''(x)$ constant. If G'' is a periodic function of time, the error in making this approximation is of the 5th order (10^{-10}) in G'' and could increase linearly with time in g''. Nevertheless, for a not too long time the error in g'' may be tolerable since it is essentially $10^{-10} n' t$. For $n' \simeq 300°/$day, the error in longitude is less than 10^{-4} degrees for several hundred days.

Therefore, if η''_0, θ''_0 and g''_0 are the values of η'', θ'' and g'' at the epoch, it is possible to consider

$$g''_1 = g''_0 - \left\{ \frac{K_1}{\eta''_0} \left[(\eta''^2_0 - 5\,\theta''^2_0) - 5\,(\eta''^2_0 - \theta''^2_0) \cos 2g''_0 \right] \right.$$

$$\left. + \frac{K_2}{L'\,\eta''^4_0} \,(1 - 5\,\theta''^2_0) \right\} t = g''_0 + n''_g\, t$$

$$G'' = G''_0 .$$

If these values are substituted in f_i ($i = 1,2,3$), the 2nd order long period perturbations in l'', g'', h'' are obtained by quadratures, i.e.

$$\Delta_2 l'' = \int f_1\,(g''_1\,(t),\, G''_0)\, dt$$

$$\Delta_2 g'' = \int f_2\,(g''_1\,(t),\, G''_0)\, dt$$

$$\Delta_2 h'' = \int f_3\,(g''_1\,(t),\, G''_0)\, dt .$$

For the variations in G'' (3rd order), the equation is

$$\Delta_2 \dot{G}'' = \frac{\partial F''_3}{\partial g''} = \mathcal{G}\,(g'',\, G'')$$

which can again be integrated considering

$$\Delta_2 G'' = \int \mathcal{G}\,(g''_1(t),\, G''_0)\, dt ,$$

where

$$g''_1\,(t) = g''_0 + n''_g\, t .$$

Note that the quadratures above will give the secular perturbations to 3rd order.

33. Additional Long Period Perturbations of Second Order and Secular Perturbations of Third Order

The third order part of the Hamiltonian, free from short periodic terms and from h'' and τ, is designated F_3'' and appears in Section 22. In the partial derivatives that follow, the Keplerian elements are considered double primed, but for convenience the primes have been dropped.

The functions f_1, f_2, f_3, and $\mathcal{G}$ are given next.

$$f_1 \, [g_1''(t), G_0''] = -\frac{\partial F_3''}{\partial L''} = -2\sqrt{\frac{a}{\mu}} \frac{\partial F_3''}{\partial a} - \frac{1-e^2}{e\sqrt{\mu a}} \frac{\partial F_3''}{\partial e}$$

$$f_2 \, [g_1''(t), G_0''] = -\frac{\partial F_3''}{\partial G''} = \frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}} \frac{\partial F_3''}{\partial e} - \frac{\cot I}{\sqrt{\mu a (1-e^2)}} \frac{\partial F_3''}{\partial I}$$

$$f_3 \, [g_1''(t), G_0''] = -\frac{\partial F_3''}{\partial H''} = \frac{1}{\sin I \sqrt{\mu a (1-e^2)}} \frac{\partial F_3''}{\partial I}$$

$$\mathcal{G} \, [g_1''(t), G_0''] = \frac{\partial F_3''}{\partial g''} ,$$

where

$$\frac{\partial F_3''}{\partial a} = +\frac{63}{256}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2} a\sqrt{1-e^2} \cos I \, \{ (2+33\,e^2) - (2-17\,e^2)\cos^2 I$$

$$+ 15\,e^2 \sin^2 I \cos 2g \}$$

$$-\frac{27}{64}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\, b^2 \frac{n^2 \sin^2 I \cos I}{a\,(1-e^2)^2\,\epsilon} \{2\,(2+3\,e^2) + 15\,e^2 \cos 2g\}$$

$$-\frac{117}{8}\left(\frac{n}{n_☾^*}\right) J_{22}^2\, b^4 \frac{n^2 \sin^2 I \cos I}{a^3\,(1-e^2)^{7/2}}$$

$$+\frac{1}{8} n_\oplus^2\, a \, \{-(2+3\,e^2)(1-3\cos^2 I) + 15\,e^2 \sin^2 I \cos 2g\}$$

$$+\frac{3}{2} J_3 b^3 \frac{n^2 e \sin I (1-5\cos^2 I)}{a^2 (1-e^2)^{5/2}} \sin g$$

$$+\frac{15}{128} J_4 b^4 \frac{n^2}{a^3 (1-e^2)^{7/2}} \{ (2+3e^2)(3-30\cos^2 I + 35\cos^4 I)$$

$$- 10 e^2 \sin^2 I (1-7\cos^2 I) \cos 2g \}$$

$$+\frac{15}{128} J_5 b^5 \frac{n^2 e \sin I}{a^4 (1-e^2)^{9/2}} \{6(4+3e^2)(1-14\cos^2 I + 21\cos^4 I) \sin g$$

$$- 7 e^2 \sin^2 I (1-9\cos^2 I) \sin 3g \}$$

$$\frac{\partial F_3''}{\partial I} = -\frac{9}{128} \left(\frac{n_☾}{n_☾^*}\right) \left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2} a^2 \sqrt{1-e^2} \sin I \{ (2+33e^2) - 3(2-17e^2)\cos^2 I$$

$$+ 15 e^2 (1-3\cos^2 I) \cos 2g\}$$

$$-\frac{9}{32} \left(\frac{n_☾}{n_☾^*}\right) \left(\frac{n_☾}{n}\right) J_{22} b^2 \frac{n^2 \sin I (1-3\cos^2 I)}{\epsilon (1-e^2)^2} \{2(2+3e^2) + 15 e^2 \cos 2g\}$$

$$-\frac{9}{4} \left(\frac{n}{n_☾^*}\right) J_{22}^2 b^4 \frac{n^2 \sin I (1-3\cos^2 I)}{a^2 (1-e^2)^{7/2}}$$

$$+\frac{3}{8} n_\oplus^2 a^2 \sin I \cos I \{ -(2+3e^2) + 5 e^2 \cos 2g\}$$

$$+\frac{3}{4} j_2 R_\oplus^2 \frac{n_☾^2 \sin I \cos I (1-3\cos^2 I_\oplus)}{\epsilon} \{1-\beta^2 (1+2\sqrt{1-e^2}) \cos 2g \}$$

$$-\frac{3}{8} J_3\, b^3 \frac{n^2\, e \cos I\, (11 - 15 \cos^2 I)}{a\, (1 - e^2)^{5/2}} \sin g$$

$$-\frac{15}{32} J_4\, b^4 \frac{n^2 \sin I \cos I}{a^2 (1 - e^2)^{7/2}} \{(2 + 3 e^2)(3 - 7 \cos^2 I) - 2 e^2 (4 - 7 \cos^2 I) \cos 2g\}$$

$$-\frac{15}{256} J_5\, b^5 \frac{n^2\, e \cos I}{a^3 (1 - e^2)^{9/2}} \{2 (4 + 3 e^2)(29 - 126 \cos^2 I + 105 \cos^4 I) \sin g$$

$$- 7 e^2 \sin^2 I\, (7 - 15 \cos^2 I) \sin 3g\}$$

$$\frac{\partial F_3''}{\partial e} = + \frac{9}{128} \left(\frac{n_{☾}}{n_{☾}^*}\right)\left(\frac{n_{☾}}{n}\right)^3 \frac{n^2\, a^2\, e \cos I}{\sqrt{1 - e^2}\, \epsilon^2} \{64 - 99 e^2 + 3 (12 - 17 e^2) \cos^2 I$$

$$+ 15\, (2 - 3 e^2) \sin^2 I \cos 2g\}$$

$$+ \frac{9}{16} \left(\frac{n_{☾}}{n_{☾}^*}\right)\left(\frac{n_{☾}}{n}\right) J_{22}\, b^2 \frac{n^2\, e \sin^2 I \cos I}{\epsilon\, (1 - e^2)^3} \{2 (7 + 3 e^2) + 15 (1 + e^2) \cos 2g\}$$

$$+ \frac{63}{4} \left(\frac{n}{n_{☾}^*}\right) J_{22}^2\, b^4 \frac{n^2\, e \sin^2 I \cos I}{a^2\, (1 - e^2)^{9/2}}$$

$$-\frac{3}{8} n_{\oplus}^2\, a^2\, e\, \{1 - 3 \cos^2 I - 5 \sin^2 I \cos 2g\}$$

$$-\frac{3}{4} j_2\, R_{\oplus}^2 \frac{n_{☾}^2 \sin^2 I\, (1 - 3 \cos^2 I_{\oplus})}{\epsilon \cdot e} \beta^2\, (2 + \sqrt{1 - e^2}) \cos 2g$$

$$- \frac{3}{8} J_3 b^3 \frac{n^2 \sin I (1 - 5 \cos^2 I)}{a (1 - e^2)^{7/2}} (1 + 4 e^2) \sin g$$

$$- \frac{15}{128} J_4 b^4 \frac{n^2 e}{a^2 (1 - e^2)^{9/2}} \{ (4 + 3 e^2) (3 - 30 \cos^2 I + 35 \cos^4 I)$$

$$- 2 (2 + 5 e^2) \sin^2 I (1 - 7 \cos^2 I) \cos 2g \}$$

$$- \frac{15}{256} J_5 b^5 \frac{n^2 \sin I}{a^3 (1 - e^2)^{11/2}} \{ 2 (4 + 41 e^2 + 18 e^4) (1 - 14 \cos^2 I + 21 \cos^4 I) \sin g$$

$$- 7 e^2 (1 + 2 e^2) \sin^2 I (1 - 9 \cos^2 I) \sin 3g \}$$

$$\frac{\partial F''_3}{\partial g} = - \frac{135}{64} \left(\frac{n_☾}{n_☾^*} \right) \left(\frac{n_☾}{n} \right)^3 \frac{n^2}{\epsilon^2} a^2 e^2 \sqrt{1 - e^2} \sin^2 I \cos I \sin 2g$$

$$- \frac{135}{16} \left(\frac{n_☾}{n_☾^*} \right) \left(\frac{n_☾}{n} \right) J_{22} b^2 \frac{n^2 e^2 \sin^2 I \cos I}{\epsilon (1 - e^2)^2} \sin 2g$$

$$- \frac{15}{8} n_\oplus^2 a^2 e^2 \sin^2 I \sin 2g$$

$$+ \frac{3}{4} j_2 R_\oplus^2 \frac{n_☾^2 \sin^2 I (1 - 3 \cos^2 I_\oplus)}{\epsilon} \beta^2 (1 + 2 \sqrt{1 - e^2}) \sin 2g$$

$$- \frac{3}{8} J_3 b^3 \frac{n^2 e \sin I (1 - 5 \cos^2 I)}{a (1 - e^2)^{5/2}} \cos g$$

$$- \frac{15}{32} J_4 b^4 \frac{n^2 e^2 \sin^2 I (1-7\cos^2 I)}{a^2 (1-e^2)^{7/2}} \sin 2g$$

$$- \frac{15}{256} J_5 b^5 \frac{n^2 e \sin I}{a^3 (1-e^2)^{9/2}} \{2 (4+3e^2)(1-14\cos^2 I + 21 \cos^4 I) \cos g$$

$$- 7 e^2 \sin^2 I (1-9\cos^2 I) \cos 3g\}.$$

Now, recalling that

$$\Delta_2 l'' = \int f_1 [g_1''(t), G_0''] dt$$

$$\Delta_2 g'' = \int f_2 [g_1''(t), G_0''] dt$$

$$\Delta_2 h'' = \int f_3 [g_1''(t), G_0''] dt$$

$$\Delta_2 G'' = \int \mathcal{G} [g_1''(t), G_0''] dt,$$

it is noted that if one performs the integrations, one obtains a small divisor (n_g'') which is of second order. The resulting perturbations will therefore be of the first order, and this is not desirable. However, since the pericenter moves slowly (about 10^{-3} rad/day), these perturbations will be of the second order if the time period of integration is of the order of hundreds of days. In the following integrals the initial values of the Keplerian elements are to be used. For convenience, however, we write $\dot{e}$ for e_0'' , etc.

$$\int_0^t \frac{\partial F_3''}{\partial a}\,dt = \Bigg\{ +\frac{63}{256}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2}\, a\,\sqrt{1-e^2}\,\cos I\,[2+33\,e^2 - (2-17\,e^2)\cos^2 I]$$

$$-\frac{27}{32}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\,b^2\,\frac{n^2\sin^2 I\cos I}{a\,(1-e^2)^2\,\epsilon}\,(2+3\,e^2)$$

$$-\frac{117}{8}\left(\frac{n}{n_☾^*}\right) J_{22}^2\,b^4\,\frac{n^2\sin^2 I\cos I}{a^3\,(1-e^2)^{7/2}} - \frac{1}{8}\,n_\oplus^2\,a\,(2+3\,e^2)\,(1-3\cos^2 I)$$

$$+\frac{15}{128}\,J_4\,b^4\,\frac{n^2}{a^3\,(1-e^2)^{7/2}}\,(3-30\cos^2 I+35\cos^4 I)\,(2+3\,e^2)\Bigg\}\,t$$

$$-\frac{1}{n_g''}\Bigg\{\frac{3}{2}\,J_3\,b^3\,\frac{n^2\,e\sin I\,(1-5\cos^2 I)}{a^2\,(1-e^2)^{5/2}}$$

$$+\frac{45}{64}\,J_5\,b^5\,\frac{n^2\,e\sin I}{a^4\,(1-e^2)^{9/2}}\,(1-14\cos^2 I+21\cos^4 I)\,(4+3\,e^2)\Bigg\}\cdot$$

$$\cdot\,(\cos g_1''-\cos g_0'')$$

$$+\frac{1}{n_g''}\Bigg\{+\frac{945}{512}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3\frac{n^2}{\epsilon^2}\,e^2\,a\,\sqrt{1-e^2}\,\cos I\sin^2 I$$

$$-\frac{405}{128}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\,b^2\,\frac{n^2\,e^2\sin^2 I\cos I}{a\,(1-e^2)^2\,\epsilon} + \frac{15}{16}\,n_\oplus^2\,a\,e^2\sin^2 I$$

$$-\frac{75}{128}\,J_4\,b^4\,\frac{n^2\,e^2}{a^3\,(1-e^2)^{7/2}}\sin^2 I\,(1-7\cos^2 I)\Bigg\}(\sin 2g_1''-\sin 2\,g_0'')$$

$$+\frac{35}{128}\,J_5\,b^5\,\frac{n^2\,e^3\sin^3 I}{a^4\,n_g''\,(1-e^2)^{9/2}}\,(1-9\cos^2 I)(\cos 3\,g_1''-\cos 3\,g_0'')$$

$$\int_0^t \frac{\partial F_3''}{\partial I}\,dt = \left\{\frac{-9}{128}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2}\, a^2 \sqrt{1-e^2}\,\sin I\,[2+33\,e^2 - 3\,(2-17\,e^2)\cos^2 I]\right.$$

$$-\frac{9}{16}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\, b^2 \frac{n^2 \sin I}{(1-e^2)^2\,\epsilon}(1-3\cos^2 I)\,(2+3\,e^2)$$

$$-\frac{9}{4}\left(\frac{n}{n_☾^*}\right) J_{22}^2\, b^4 \frac{n^2 \sin I}{a^2\,(1-e^2)^{7/2}}(1-3\cos^2 I)$$

$$-\frac{3}{8}\,n_\oplus^2\, a^2 \sin I \cos I\,(2+3e^2)$$

$$+\frac{3}{4}\, j_2\, R_\oplus^2 \frac{n_☾^2 \sin I \cos I}{\epsilon}(1-3\cos^2 I_\oplus)$$

$$\left.-\frac{15}{32} J_4\, b^4 \frac{n^2 \sin I \cos I}{a^2\,(1-e^2)^{7/2}}(3-7\cos^2 I)\,(2+3\,e^2)\right\} t$$

$$+\frac{1}{n_g''}\left\{\frac{3}{8} J_3\, b^3 \frac{n^2\, e \cos I}{a\,(1-e^2)^{5/2}}(11-15\cos^2 I)\right.$$

$$\left.+\frac{15}{128} J_5\, b^5 \frac{n^2\, e\cos I}{a^3\,(1-e^2)^{9/2}}(29-126\cos^2 I+105\cos^4 I)\,(4+3\,e^2)\right\}\cdot$$

$$\cdot\,(\cos g_1'' - \cos g_0'')$$

$$+\frac{1}{n_g''}\left\{\frac{-135}{256}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2}\, a^2\, e^2 \sqrt{1-e^2}\,\sin I\,(1-3\cos^2 I)\right.$$

$$-\frac{135}{64}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\, b^2 \frac{n^2\, e^2 \sin I}{(1-e^2)^2\,\epsilon}(1-3\cos^2 I)$$

$$+\frac{15}{16}\,n_\oplus^2\, a^2\, e^2 \sin I \cos I$$

$$+\frac{15}{32} J_4 b^4 \frac{n^2 e^2 \sin I \cos I}{a^2 (1-e^2)^{7/2}} (4 - 7 \cos^2 I)$$

$$\left. -\frac{3}{8} j_2 R_\oplus^2 \frac{n_☾^2 \sin I \cos I (1-3\cos^2 I_\oplus) \beta^2 (1+2\sqrt{1-e^2})}{\epsilon} \right\} \cdot$$

$$\cdot (\sin 2g_1'' - \sin 2 g_0'')$$

$$-\frac{35}{256} J_5 b^5 \frac{n^2 e^3 \cos I}{n_g'' a^3 (1-e^2)^{9/2}} \sin^2 I (7 - 15 \cos^2 I)(\cos 3 g_1'' - \cos 3 g_0'')$$

$$\int_0^t \frac{\partial F_3''}{\partial e} dt = + \left\{ \frac{9}{128} \left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2 a^2 e \cos I}{(1-e^2)^{1/2} \epsilon^2} [64 - 99 e^2 + 3 (12 - 17 e^2) \cos^2 I] \right.$$

$$+\frac{9}{8}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22} b^2 \frac{n^2 e}{(1-e^2)^3 \epsilon} \sin^2 I \cos I (7 + 3 e^2)$$

$$+\frac{63}{4}\left(\frac{n}{n_☾^*}\right) J_{22}^2 b^4 \frac{n^2 e}{a^2 (1-e^2)^{9/2}} \sin^2 I \cos I - \frac{3}{8} n_\oplus^2 a^2 e (1 - 3\cos^2 I)$$

$$\left. -\frac{15}{128} J_4 b^4 \frac{n^2 e}{a^2 (1-e^2)^{9/2}} (3 - 30 \cos^2 I + 35 \cos^4 I)(4 + 3 e^2) \right\} t$$

$$+\frac{1}{n_g''} \left\{ \frac{3}{8} J_3 b^3 \frac{n^2 \sin I}{a (1-e^2)^{7/2}} (1 - 5\cos^2 I)(1 + 4 e^2) \right.$$

$$\left. +\frac{15}{128} J_5 b^5 \frac{n^2 \sin I}{a^3 (1-e^2)^{11/2}} (1 - 14 \cos^2 I + 21 \cos^4 I)(4 + 41 e^2 + 18 e^4) \right\} \cdot$$

$$\cdot (\cos g_1'' - \cos g_0'')$$

$$+\frac{1}{n_g''}\left\{+\frac{135}{256}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2 a^2 e}{\epsilon^2 (1-e^2)^{1/2}} \cos I \sin^2 I\, (2-3e^2)\right.$$

$$+\frac{135}{32}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\, b^2 \frac{n^2 e}{\epsilon\,(1-e^2)^3} \sin^2 I \cos I\,(1+e^2) + \frac{15}{16} n_\oplus^2\, a^2 e \sin^2 I$$

$$-\frac{3}{8} j_2 R_\oplus^2 \frac{n_☾^2}{\epsilon \cdot e} \sin^2 I\,(1-3\cos^2 I_\oplus)\,\beta^2\,(2+\sqrt{1-e^2})$$

$$\left.+\frac{15}{128} J_4\, b^4 \frac{n^2 e}{a^2(1-e^2)^{9/2}} \sin^2 I\,(1-7\cos^2 I)(2+5e^2)\right\}(\sin 2g_1'' - \sin 2g_0'')$$

$$-\frac{35}{256} J_5\, b^5 \frac{n^2 e^2}{n_g''\, a^3\,(1-e^2)^{11/2}} \sin^3 I\,(1-9\cos^2 I)\,(1+2e^2)(\cos 3g_1'' - \cos 3g_0'')$$

$$\int_0^t \frac{\partial F_3''}{\partial g}\, dt = -\frac{1}{n_g''}\left\{\frac{3}{8} J_3\, b^3 \frac{n^2 e \sin I}{a\,(1-e^2)^{5/2}} (1-5\cos^2 I) + \frac{15}{128} J_5\, b^5 \frac{n^2 e \sin I}{a^3 (1-e^2)^{9/2}} \cdot\right.$$

$$\left.\cdot\,(1-14\cos^2 I + 21\cos^4 I)\,(4+3e^2)\right\}(\sin g_1'' - \sin g_0'')$$

$$+\frac{1}{n_g''}\left\{+\frac{135}{128}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right)^3 \frac{n^2}{\epsilon^2} a^2 e^2 \sqrt{1-e^2} \sin^2 I \cos I\right.$$

$$+\frac{135}{32}\left(\frac{n_☾}{n_☾^*}\right)\left(\frac{n_☾}{n}\right) J_{22}\, b^2 \frac{n^2 e^2}{\epsilon\,(1-e^2)^2} \sin^2 I \cos I$$

$$+\frac{15}{16} n_\oplus^2\, a^2 e^2 \sin^2 I$$

$$-\frac{3}{8} j_2 R_\oplus^2 \frac{n_☾^2}{\epsilon} \sin^2 I\,(1-3\cos^2 I_\oplus)\,\beta^2\,(1+2\sqrt{1-e^2})$$

$$+\frac{15}{64} J_4 b^4 \frac{n^2 e^2}{a^2 (1-e^2)^{7/2}} \sin^2 I (1-7\cos^2 I) \Big\}(\cos 2g_1'' - \cos 2g_0'')$$

$$+\frac{35}{256} J_5 b^5 \frac{n^2 e^3}{n_g'' a^3 (1-e^2)^{9/2}} \sin^3 I (1-9\cos^2 I)(\sin 3g_1'' - \sin 3g_0'')$$

34. Summary of the Development

The short-period terms are given by

$$l = l' - \frac{\partial S_2}{\partial L'}$$

$$g = g' - \frac{\partial S_2}{\partial G'}$$

$$h = h' - \frac{\partial S_2}{\partial H'}$$

$$L = L' + \frac{\partial S_2}{\partial l}$$

$$G = G' + \frac{\partial S_2}{\partial g}$$

$$H = H' + \frac{\partial S_2}{\partial h} .$$

Long-period terms are given by

$$l' = l'' - \frac{\partial S_1'}{\partial L''} - \frac{\partial S_2'}{\partial L''} - \frac{\partial S_3'}{\partial L''}$$

$$g' = g'' - \frac{\partial S_1'}{\partial G''} - \frac{\partial S_2'}{\partial G''} - \frac{\partial S_3'}{\partial G''}$$

$$h' = h'' - \frac{\partial S_1'}{\partial H''} - \frac{\partial S_2'}{\partial H''} - \frac{\partial S_3'}{\partial H''}$$

$$L' = L''$$

$$G' = G'' + \frac{\partial S_1'}{\partial g'} + \frac{\partial S_2'}{\partial g'} + \frac{\partial S_3'}{\partial g'}$$

$$H' = H'' + \frac{\partial S_1'}{\partial h'} + \frac{\partial S_2'}{\partial h'} + \frac{\partial S_3'}{\partial h'} .$$

Additional terms are obtained from

$$\Delta_1 \ell'' = \int \left(n' - \frac{\partial F_2''}{\partial L''} \right) dt$$

$$\Delta_2 \ell'' = - \int \frac{\partial F_3''}{\partial L''} \, dt$$

$$\ell'' - \ell_0'' = \Delta_1 \ell'' + \Delta_2 \ell''$$

$$\Delta_1 g'' = - \int \frac{\partial F_2''}{\partial G''} \, dt$$

$$\Delta_2 g'' = - \int \frac{\partial F_3''}{\partial G''} \, dt$$

$$g'' - g_0'' = \Delta_1 g'' + \Delta_2 g''$$

$$\Delta_1 h'' = -\int \left(n_{\mathbb{C}}^* + \frac{\partial F_2''}{\partial H''} \right) dt$$

$$\Delta_2 h'' = -\int \frac{\partial F_3''}{\partial H''} \, dt$$

$$h'' - h_0'' = \Delta_1 h'' + \Delta_2 h''$$

$$L'' - L_0'' = 0$$

$$\Delta_1 G'' = \int \frac{\partial F_2''}{\partial g} \, dt$$

$$\Delta_2 G'' = \int \frac{\partial F_3''}{\partial g} \, dt$$

$$G'' - G_0'' = \Delta_1 G'' + \Delta_2 G''$$

$$H'' - H_0'' = 0 .$$

Then we have

$\Delta L = L - L'$	$\Delta L' = 0$	$\Delta L'' = 0$
$\Delta G = G - G'$	$\Delta G' = G' - G''$	$\Delta G'' = G'' - G_0''$
$\Delta H = H - H'$	$\Delta H' = H' - H''$, and	$\Delta H'' = 0$.

Note that in the expressions for $\Delta_1 G$, $\Delta_1 \ell$, $\Delta_1 g$, and $\Delta_1 h$ the variables $G_0'' + \Delta_2 G''$ and $g_0'' + \Delta_2 g''$ should be used in computing ν^2, θ''^2, C, etc. Thus, the perturbations in the Keplerian elements a, e, and I are obtained by

$$a = a' + 2a' \frac{\Delta L}{L'}$$

$$e = e' + \frac{1 - e'^2}{e'} \left(\frac{\Delta L}{L'} - \frac{\Delta G}{G'} \right)$$

$$I = I' + \cot I' \left(\frac{\Delta G}{G'} - \frac{\Delta H}{H'} \right).$$

Also,

$$a' = a''$$

$$e' = e'' - \frac{1 - e''^2}{e''} \frac{\Delta G'}{G''} - \frac{1 - e''^2}{2 e''^3} \left(\frac{\Delta G'}{G''} \right)^2$$

$$I' = I'' + \cot I'' \left(\frac{\Delta G'}{G''} - \frac{\Delta H'}{H''} \right) + \frac{\cos I''}{\sin^3 I''} \left(\frac{\Delta G'}{G''} \right) \left(\frac{\Delta H'}{H''} \right) - \frac{1}{2} \frac{\cos I''}{\sin^3 I''} (1 + \sin^2 I'') \left(\frac{\Delta G'}{G''} \right)^2$$

$$- \frac{1}{2} \frac{\cos^3 I''}{\sin^3 I''} \left(\frac{\Delta H'}{H''} \right)^2,$$

where in the squares and products of $\Delta G'$ and $\Delta H'$ terms of the 3rd order or superior should be neglected. Finally,

$$a'' = a''_0$$

$$e'' = e''_0 - \frac{1 - e''^2_0}{e''_0} \frac{\Delta G''}{G''_0}$$

$$I'' = I''_0 + \cot I''_0 \frac{\Delta G''}{G''_0}.$$

35. Position and Velocity

$e \neq 0;\ I \neq 0°, 180°$

From the elements L, G, H, l, g, h, one can obtain the coordinates and the components of velocity as follows:

Obtain a, e, and I from the following equations:

$$a = L^2/\mu$$

$$e = +\sqrt{1 - G^2/L^2}$$

$$I = \arc\cos \frac{H}{G} \quad (0^\circ < I < 180^\circ).$$

Now solve Kepler's equation

$$E - e \sin E = l$$

to obtain E.

Then compute r from

$$r = a(1 - e \cos E).$$

Next obtain f from

$$\cos f = \frac{a}{r}(\cos E - e)$$

$$\sin f = \frac{a}{r}\frac{G}{L}\sin E.$$

Now compute

$$A_x = a\,[\cos g \cos h - \sin g \sin h \cos I]$$

$$B_x = a\frac{G}{L}\,[-\sin g \cos h - \cos g \sin h \cos I]$$

$$A_y = a\,[\cos g \sin h + \sin g \cos h \cos I]$$

$$B_y = a\frac{G}{L}\,[-\sin g \sin h + \cos g \cos h \cos I]$$

$$A_z = a \sin g \sin I$$

$$B_z = a\frac{G}{L}\cos g \sin I.$$

Then,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A_x & B_x & 0 \\ A_y & B_y & 0 \\ A_z & B_z & 0 \end{pmatrix} \begin{pmatrix} \cos E - e \\ \sin E \\ 0 \end{pmatrix} .$$

Since the system is rotating,

$$\dot{A}_x = n_{☾}^{*} A_y$$

$$\dot{B}_x = n_{☾}^{*} B_y$$

$$\dot{A}_y = - n_{☾}^{*} A_x$$

$$\dot{B}_y = - n_{☾}^{*} B_x .$$

Therefore,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = n_{☾}^{*} \begin{pmatrix} A_y & B_y & 0 \\ -A_x & -B_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos E - e \\ \sin E \\ 0 \end{pmatrix}$$

$$+ \frac{a}{r} n \begin{pmatrix} A_x & B_x & 0 \\ A_y & B_y & 0 \\ A_z & B_z & 0 \end{pmatrix} \begin{pmatrix} - \sin E \\ \cos E \\ 0 \end{pmatrix} .$$

36. References and Bibliography

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37. Appendix A

The elliptic integrals that are needed for this section may be found in Reference 3. They are

1. $$\int_{x_1'}^{x} \frac{dx}{(x - \nu^2)\sqrt{Q(x)}}$$

2. $$\int_{x_1'}^{x} \frac{dx}{x\sqrt{Q(x)}}$$

3. $$\int_{x_1'}^{x} \frac{dx}{(1 - x)\sqrt{Q(x)}}$$

4. $$\int_{x_1'}^{x} \frac{dx}{x^2\sqrt{Q(x)}}$$

5. $$\int_{x_1'}^{x} \frac{dx}{x^3\sqrt{Q(x)}}$$

6. $$\int_{x_1'}^{x} \frac{dx}{(x - \nu^2)\sqrt{xQ(x)}}$$

7. $$\int_{x_1'}^{x} \frac{dx}{(1 - x)\sqrt{xQ(x)}}$$

8. $$\int_{x_1'}^{x} \frac{dx}{\sqrt{xQ(x)}}$$

9. $$\int_{x_1'}^{x} \frac{dx}{x\sqrt{xQ(x)}}$$

10. $$\int_{x_1'}^{x} \frac{dx}{x^2\sqrt{xQ(x)}},$$

where $Q(x) = (x - x_1')\ (x - x_2')\ (x - x_3')$, and $x_3' > x_2' \geq x > x_1' > 0$.

<u>1-5</u>

Let

$$\sin^2\psi = \frac{x - x_1'}{x_2' - x_1'},\quad k^2 = \frac{x_2' - x_1'}{x_3' - x_1'}.$$

We then have

$$\int_{x_1'}^{x} \frac{dx}{(p - x)^m \sqrt{Q(x)}} = \frac{2}{(p - x_1')^m \sqrt{x_3' - x_1'}} \int_0^u \frac{du}{(1 - \alpha^2 \operatorname{sn}^2 u)^m},$$

where

$$\operatorname{sn} u = \sin \psi, \quad \alpha^2 = \frac{x_2' - x_1'}{p - x_1'}.$$

For m = 1:

$$\int_{x_1'}^{x} \frac{dx}{(p - x) \sqrt{Q(x)}} = \frac{2}{(p - x_1') \sqrt{x_3' - x_1'}} \int_0^u \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u}$$

$$= \frac{2}{(p - x_1') \sqrt{x_3' - x_1'}} \Pi(\psi, \alpha^2, k).$$

Thus,

$$\int_{x_1'}^{x} \frac{dx}{(x - \nu^2) \sqrt{Q(x)}} = \frac{2}{(x_1' - \nu^2) \sqrt{x_3' - x_1'}} \Pi(\psi, \alpha^2, k), \qquad \alpha^2 = \frac{x_2' - x_1'}{\nu^2 - x_1'}$$

$$\int_{x_1'}^{x} \frac{dx}{x \sqrt{Q(x)}} = \frac{2}{x_1' \sqrt{x_3' - x_1'}} \Pi(\psi, \alpha^2, k), \qquad \alpha^2 = -\frac{x_2' - x_1'}{x_1'}$$

$$\int_{x_1'}^{x} \frac{dx}{(1 - x) \sqrt{Q(x)}} = \frac{2}{(1 - x_1') \sqrt{x_3' - x_1'}} \Pi(\psi, \alpha^2, k), \qquad \alpha^2 = \frac{x_2' - x_1'}{1 - x_1'}.$$

For m = 2:

$$\int_{x_1'}^{x} \frac{dx}{x^2\sqrt{Q(x)}} = \frac{2}{x_1'^2\sqrt{x_3'-x_1'}} \int_0^u \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}, \qquad \alpha^2 = -\frac{x_2'-x_1'}{x_1'}$$

$$= \frac{2}{x_1'^2\sqrt{x_3'-x_1'}}\, V_2,$$

where

$$V_2 = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + (2\,\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,\Pi(\psi,\alpha^2,k) - \frac{\alpha^4\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-\alpha^2\,\mathrm{sn}^2 u}\right].$$

where $u = F(\psi, k)$

and $E(u) = E(\psi, k)$

For: m = 3:

$$\int_{x_1'}^{x} \frac{dx}{x^3\sqrt{Q(x)}} = \frac{2}{x_1'^3\sqrt{x_3'-x_1'}} \int_0^u \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}, \qquad \alpha^2 = -\frac{x_2'-x_1'}{x_1'}$$

$$= \frac{2}{x_1'^3\sqrt{x_3'-x_1'}}\, V_3,$$

where

$$V_3 = \frac{1}{4(1-\alpha^2)(k^2-\alpha^2)}\left[k^2 u + 2(\alpha^2 k^2 + \alpha^2 - 3k^2)\,\Pi(\psi,\alpha^2,k) + 3(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2)\,V_2 + \frac{\alpha^4\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}\right].$$

6 & 7

Let

$$\sin^2 \psi = \frac{x_2'}{x_2' - x_1'} \cdot \frac{x - x_1'}{x}, \quad k^2 = \frac{x_3'(x_2' - x_1')}{x_2'(x_3' - x_1')}, \quad \alpha^2 = \frac{x_2' - x_1'}{x_2'}.$$

We have

$$\int_{x_1'}^{x} \frac{dx}{(p - x)^m \sqrt{xQ(x)}} = \frac{2}{(p - x_1')^m \sqrt{x_2'(x_3' - x_1')}} \int_0^u \frac{(1 - \alpha^2 \operatorname{sn}^2 u)^m}{(1 - \alpha_3^2 \operatorname{sn}^2 u)^m} du,$$

where

$$\operatorname{sn} u = \sin \psi, \quad \alpha_3^2 = \frac{p(x_2' - x_1')}{x_2'(p - x_1')}.$$

For $m = 1$, $p \neq 0$:

$$\int_{x_1'}^{x} \frac{dx}{(p - x) \sqrt{xQ(x)}} = \frac{2}{(p - x_1') \sqrt{x_2'(x_3' - x_1')}} \int_0^u \frac{1 - \alpha^2 \operatorname{sn}^2 u}{1 - \alpha_3^2 \operatorname{sn}^2 u} du$$

$$= \frac{2}{p\sqrt{x_2'(x_3' - x_1')}} \left[u + \frac{x_1'}{p - x_1'} \Pi(\psi, \alpha_3^2, k) \right].$$

Thus,

$$\int_{x_1'}^{x} \frac{dx}{(x - \nu^2)\sqrt{xQ(x)}} = \frac{-2}{\nu^2\sqrt{x_2'(x_3' - x_1')}} \left[u + \frac{x_1'}{\nu^2 - x_1'} \Pi(\psi, \alpha_3^2, k) \right], \quad \alpha_3^2 = \frac{\nu^2(x_2' - x_1')}{x_2'(\nu^2 - x_1')}$$

$$\int_{x_1'}^{x} \frac{dx}{(1-x)\sqrt{xQ(x)}} = \frac{2}{\sqrt{x_2'(x_3'-x_1')}} \left[u + \frac{x_1'}{1-x_1'} \Pi(\psi, \alpha_3^2, k)\right], \quad \alpha_3^2 = \frac{x_2'-x_1'}{x_2'(1-x_1')}.$$

8-10

Again, let

$$\sin^2\psi = \frac{x_2'}{x_2'-x_1'} \cdot \frac{x-x_1'}{x}, \quad k^2 = \frac{x_3'(x_2'-x_1')}{x_2'(x_3'-x_1')}, \quad \alpha^2 = \frac{x_2'-x_1'}{x_2'}.$$

We have

$$\int_{x_1'}^{x} \frac{dx}{x^m\sqrt{xQ(x)}} = \frac{2}{x_1'^m\sqrt{x_2'(x_3'-x_1')}} \int_0^u (1-\alpha^2 \operatorname{sn}^2 u)^m \, du.$$

For $m = 0$:

$$\int_{x_1'}^{x} \frac{dx}{\sqrt{xQ(x)}} = \frac{2}{\sqrt{x_2'(x_3'-x_1')}} u.$$

For $m = 1$:

$$\int_{x_1'}^{x} \frac{dx}{x\sqrt{xQ(x)}} = \frac{2}{x_1'\sqrt{x_2'(x_3'-x_1')}} \int_0^u (1-\alpha^2 \operatorname{sn}^2 u)\, du$$

$$= \frac{2}{x_1'\sqrt{x_2'(x_3'-x_1')}} \cdot \frac{1}{k^2}\left[(k^2-\alpha^2)u + \alpha^2 E(u)\right]$$

$$= \frac{2}{x_1' x_3' \sqrt{x_2'(x_3'-x_1')}} \left[x_1' u + (x_3'-x_1') E(u)\right].$$

For m = 2:

$$\int_{x_1'}^{x} \frac{dx}{x^2 \sqrt{xQ(x)}} = \frac{2}{x_1'^2 \sqrt{x_2' (x_3' - x_1')}} \int_0^u (1 - \alpha^2 \operatorname{sn}^2 u)^2 \, du$$

$$= \frac{2}{x_1'^2 \sqrt{x_2' (x_3' - x_1')}} \cdot \frac{1}{3k^4} \Big[(3k^4 - 6\alpha^2 k^2 + 2\alpha^4 + k^2\alpha^4) u$$

$$+ 2(3\alpha^2 k^2 - \alpha^4 - k^2 \alpha^4) E(u) + \alpha^4 k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\Big].$$

38. Appendix B

The integrals that are needed for this section may be found in Reference 8. They are

1. $$\int \frac{dx}{(x - \nu^2)(x - 1)\sqrt{x - x_3}}$$

2. $$\int \frac{dx}{x(x - 1)\sqrt{x - x_3}}$$

3. $$\int \frac{dx}{(1 - x)^2\sqrt{x - x_3}}$$

4. $$\int \frac{dx}{(x - \nu^2)(x - 1)\sqrt{x(x - x_3)}}$$

5. $$\int \frac{dx}{x(x - 1)\sqrt{x(x - x_3)}}$$

6. $$\int \frac{dx}{(x - 1)\sqrt{x(x - x_3)}}$$

7. $$\int \frac{dx}{x^2(x - 1)\sqrt{x - x_3}}$$

8. $$\int \frac{dx}{x^3(x - 1)\sqrt{x - x_3}}$$

9. $$\int \frac{dx}{(1 - x)^2\sqrt{x(x - x_3)}}$$

10. $$\int \frac{dx}{x^2(x - 1)\sqrt{x(x - x_3)}}$$

1.

$$\int \frac{dx}{(x - \nu^2)(x - 1)\sqrt{x - x_3}} = \frac{1}{\nu^2 - 1}\left\{\int \frac{dx}{(x - \nu^2)\sqrt{x - x_3}} + \int \frac{dx}{(1 - x)\sqrt{x - x_3}}\right\}$$

$$\int \frac{dx}{(x - \nu^2)\sqrt{x - x_3}} = \frac{1}{\sqrt{\nu^2 - x_3}} \log \frac{\sqrt{x - x_3} - \sqrt{\nu^2 - x_3}}{\sqrt{x - x_3} + \sqrt{\nu^2 - x_3}} \quad (\nu^2 > x_3)$$

$$= \frac{2}{\sqrt{x_3 - \nu^2}} \tan^{-1} \frac{\sqrt{x - x_3}}{\sqrt{x_3 - \nu^2}} \quad (x_3 > \nu^2)$$

and

$$\int \frac{dx}{(1 - x)\sqrt{x - x_3}} = \frac{1}{\sqrt{1 - x_3}} \log \frac{\sqrt{x - x_3} + \sqrt{1 - x_3}}{\sqrt{1 - x_3} - \sqrt{x - x_3}}$$

2.

$$\int \frac{dx}{x(x - 1)\sqrt{x - x_3}} = -\int \frac{dx}{x\sqrt{x - x_3}} - \int \frac{dx}{(1 - x)\sqrt{x - x_3}}$$

$$\int \frac{dx}{x\sqrt{x - x_3}} = \frac{2}{\sqrt{x_3}} \tan^{-1} \frac{\sqrt{x - x_3}}{\sqrt{x_3}}$$

The second of these integrals may be found in Part 1.

3.

$$\int \frac{dx}{(1 - x)^2\sqrt{x - x_3}} = \frac{1}{1 - x_3}\left\{\frac{\sqrt{x - x_3}}{1 - x} + \frac{1}{2\sqrt{1 - x_3}} \log \frac{\sqrt{x - x_3} + \sqrt{1 - x_3}}{\sqrt{1 - x_3} - \sqrt{x - x_3}}\right\}$$

4.

$$\int \frac{dx}{(x - \nu^2)(x - 1)\sqrt{x(x - x_3)}} = \frac{1}{\nu^2 - 1}\left\{\int \frac{dx}{(x - \nu^2)\sqrt{x(x - x_3)}} + \int \frac{dx}{(1 - x)\sqrt{x(x - x_3)}}\right\}$$

$$\int \frac{dx}{(x - \nu^2)\sqrt{x(x-x_3)}} = \frac{1}{\sqrt{(\nu^2-x_3)\nu^2}} \log\left[\frac{2\nu^2(\nu^2 - x_3) + (2\nu^2 - x_3)(x - \nu^2) - 2\sqrt{x(x-x_3)(\nu^2-x_3)\nu^2}}{x - \nu^2}\right] (\nu^2 > x_3)$$

$$\int \frac{dx}{(x - \nu^2)\sqrt{x(x-x_3)}} = \frac{1}{\sqrt{(x_3-\nu^2)\nu^2}} \tan^{-1}\left[\frac{-2\nu^2(x_3 - \nu^2) + (2\nu^2 - x_3)(x - \nu^2)}{2\sqrt{x(x - x_3)(x_3 - \nu^2)\nu^2}}\right] (\nu^2 < x_3)$$

$$\int \frac{dx}{(1 - x)\sqrt{x(x - x_3)}} = \frac{1}{\sqrt{1 - x_3}} \log\left[\frac{2(1 - x_3) + (x_3 - 2)(1 - x) + 2\sqrt{x(x - x_3)(1 - x_3)}}{1 - x}\right]$$

5.

$$\int \frac{dx}{x(x - 1)\sqrt{x(x - x_3)}} = - \int \frac{dx}{x\sqrt{x(x - x_3)}} - \int \frac{dx}{(1 - x)\sqrt{x(x - x_3)}}$$

$$\int \frac{dx}{x\sqrt{x(x - x_3)}} = 2\,\frac{\sqrt{x(x - x_3)}}{x_3 x}$$

The second of these integrals may be found in Part 4.

6.

$$\int \frac{dx}{(x - 1)\sqrt{x(x - x_3)}} = -\frac{1}{\sqrt{1 - x_3}} \log\left[\frac{2(1 - x_3) + (x_3 - 2)(1 - x) + 2\sqrt{x(x - x_3)(1 - x_3)}}{1 - x}\right]$$

7.

$$\int \frac{dx}{x^2(x - 1)\sqrt{x - x_3}} = - \int \frac{dx}{x^2\sqrt{x - x_3}} - \int \frac{dx}{x\sqrt{x - x_3}} + \int \frac{dx}{(x - 1)\sqrt{x - x_3}}$$

$$\int \frac{dx}{x^2\sqrt{x - x_3}} = \frac{\sqrt{x - x_3}}{x_3 x} + \frac{1}{2x_3} \int \frac{dx}{x\sqrt{x - x_3}}$$

The integrals

$$\int \frac{dx}{(x-1)\sqrt{x-x_3}} \quad \text{and} \quad \int \frac{dx}{x\sqrt{x-x_3}}$$

may be found in Parts 1 and 2, respectively.

8.

$$\int \frac{dx}{x^3(x-1)\sqrt{x-x_3}} = -\int \frac{dx}{x^3\sqrt{x-x_3}} - \int \frac{dx}{x^2\sqrt{x-x_3}} - \int \frac{dx}{x\sqrt{x-x_3}} + \int \frac{dx}{(x-1)\sqrt{x-x_3}}$$

$$\int \frac{dx}{x^3\sqrt{x-x_3}} = \frac{\sqrt{x-x_3}}{2x_3x^2} + \frac{3}{4x_3} \int \frac{dx}{x^2\sqrt{x-x_3}}$$

The integrals

$$\int \frac{dx}{x^2\sqrt{x-x_3}}, \quad \int \frac{dx}{(x-1)\sqrt{x-x_3}} \quad \text{and} \quad \int \frac{dx}{x\sqrt{x-x_3}}$$

may be found in Parts 7, 1, and 2, respectively.

9.

$$\int \frac{dx}{(1-x)^2\sqrt{x(x-x_3)}} = \frac{\sqrt{x(x-x_3)}}{(1-x_3)(1-x)} + \frac{2-x_3}{2(1-x_3)} \int \frac{dx}{(1-x)\sqrt{x(x-x_3)}}$$

The integral

$$\int \frac{dx}{(1 - x)\sqrt{x(x - x_3)}}$$

may be found in Part 4.

10.

$$\int \frac{dx}{x^2(x - 1)\sqrt{x(x - x_3)}} = -\int \frac{dx}{x^2\sqrt{x(x - x_3)}} - \int \frac{dx}{x\sqrt{x(x - x_3)}} + \int \frac{dx}{(x - 1)\sqrt{x(x - x_3)}}$$

$$\int \frac{dx}{x^2\sqrt{x(x - x_3)}} = \frac{2}{3}\left(\frac{2}{x_3^2} + \frac{1}{xx_3}\right)\sqrt{1 - \frac{x_3}{x}}$$

The integrals

$$\int \frac{dx}{x\sqrt{x(x - x_3)}} \quad \text{and} \quad \int \frac{dx}{(x - 1)\sqrt{x(x - x_3)}}$$

may be found in Parts 5 and 6, respectively.